A SYSTEM

OF

PLANE AND SPHERICAL

TRIGONOMETRY;

TO WHICH IS ADDED A

TREATISE ON LOGARITHMS.

BY

THE REV. RICHARD WILSON, M.A.

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

Hoc genus in rebus firmandum est multa, prius quam
Ipsius rei rationem reddere possis,
Et nimium longis ambagibus est adeundum:
Quo magis attentas auris, animumque reposco.—LUCRET.

CAMBRIDGE:

PRINTED FOR

J. DEIGHTON, T. STEVENSON, AND R. NEWB

AND WHITTAKER, TREACHER AND CO., LONDON.

1831.

536

A SYSTEM

OF

PLANE AND SPHERICAL

TRIGONOMETRY;

TO WHICH IS ADDED A

TREATISE ON LOGARITHMS.

BY

THE REV. RICHARD WILSON, M.A.

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

Hoc genus in rebus firmandum est multa, prius quam
Ipsius rei rationem reddere possis,
Et nimium longis ambagibus est adeundum:
Quo magis attentas aureis, animumque reposco.——LUCRET.

CAMBRIDGE:

PRINTED FOR

J. & J. J. DEIGHTON, T. STEVENSON, AND R. NEWBY;

AND WHITTAKER, TREACHER AND CO., LONDON.

1831.

CAMBRIDGE:
PRINTED BY W. METCALFE, ST. MARY'S STREET.

TO

W. CAVENDISH, ESQ., M. A.,

M. P. for the University of Cambridge,

WITH FEELINGS OF THE HIGHEST RESPECT,

AND IN ADMIRATION OF THE TALENTS

SO CONSPICUOUSLY DISPLAYED BY HIM,

WHILST A STUDENT IN THE UNIVERSITY,

THIS TREATISE

IS INSCRIBED

BY HIS OBEDIENT SERVANT,

THE AUTHOR.

PREFACE.

In offering the present work to public notice, the Author feels compelled to acknowledge the obligation under which he lies, for the assistance of several of his friends in the University. In particular, he takes the opportunity of publicly returning thanks to Mr. Stephenson, Fellow of St. John's College, for many valuable papers in Plane Trigonometry. In the chapter on Multiple Arcs, considerable extracts have been made from Poinsot's *Sections Angulaires*. The matter, however, has been modified, so as to be more easily understood by the beginner, and some of the proofs have been considerably simplified. An article on Trigonometry, in the Encyclopædia Metropolitana, (written by Professor Airy,) contains many theorems of the greatest practical utility: the Author has been considerably assisted by the chapter on Geodetic Operations, in that article.

Several errors of the press have been overlooked; but, as the Author was absent from Cambridge during the time of printing, he is happy to find that, from the attention of the Printer, they are neither so many, nor so important, as might have been expected.

LIVERPOOL,
1st January, 1831.

CONTENTS.

PART I.

PLANE TRIGONOMETRY.

SECTION I.

ON ANGLES AND TRIGONOMETRIC FUNCTIONS.

SECTION II.

ON THE TRIGONOMETRIC FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ARCS.

SECTION III.

ON THE SOLUTION OF PLANE TRIANGLES.

SECTION IV.

ON MULTIPLE ARCS.

SECTION II.

ON SPHERICAL TRIANGLES.

SECTION III.

ON THE INVESTIGATION OF SUCH EQUATIONS AND PROPERTIES AS ARE NECESSARY FOR THE SOLUTION OF SPHERICAL TRIANGLES.

SECTION IV.

ON THE APPLICATION OF THE EQUATIONS ESTABLISHED IN THE PRECEDING SECTION TO THE SOLUTION OF SPHERICAL TRIANGLES.

SECTION V.

ON THE REMARKABLE REPRESENTATIONS OF THE SPHERICAL EXCESS, AND OTHER INTERESTING PROBLEMS.

SECTION VI.

ON THE MEASURING OF SOLID ANGLES.

SECTION VII.

ON THE SMALL CORRESPONDING VARIATIONS OF THE PARTS OF A TRIANGLE.

SECTION VIII.

ON GEODETIC OPERATIONS.

SECTION IX.

ON REGULAR POLYHEDRONS, &c.

APPENDIX.

A SYSTEM

OF

PLANE AND SPHERICAL TRIGONOMETRY.

PART I.

PLANE TRIGONOMETRY.

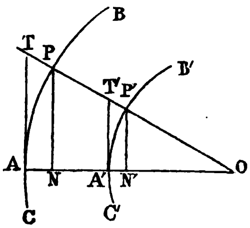
SECTION I.

ON ANGLES AND THEIR TRIGONOMETRIC FUNCTIONS.

1. **Lemma.** *The circumferences of circles are to one another as the radii.*

Let ABC and A′B′C′ be two circles, of which the radii are r and r'.

Place them so that their centres may coincide at O; draw two radii cutting the circles in A, P, A′, P′; from P, P′ draw PN, P′N′ perpendiculars to OA; and from A, A′ draw AT, A′T′ at right angles to AO, meeting OP, OP′ produced in T, T′; therefore the four triangles TAO, T′A′O, PNO, P′N′O, have each a right angle, and the angle at O common, therefore the remaining angles are equal, and the triangles are similar:



Now $\dfrac{\text{circumference ABC}}{\text{four right angles}} = \dfrac{\text{arc AP}}{\text{angle at O}}$; (Euc. VI. 33.)

And $\dfrac{\text{circumference A}'\text{B}'\text{C}'}{\text{four right angles}} = \dfrac{\text{arc A}'\text{P}'}{\text{angle at O}}$;

$\therefore \dfrac{\text{circumference ABC}}{\text{circumference A}'\text{B}'\text{C}'} = \dfrac{\text{arc AP}}{\text{arc A}'\text{P}'}$;

this will remain true, whatever be the magnitude of the angle at O; let it then be diminished without limit, and $\frac{AT}{PN}$, which by similar triangles is equal to $\frac{AO}{NO}$, in the vanishing state, becomes $\frac{AO}{AO}$, or 1; that is, in this state AT coincides with PN. But it is evident, that AP is not greater than AT, and not less than PN; hence, in the vanishing state, it coincides with either of them. Similarly A'P' coincides either with A'T' or P'N':

$\therefore$ in the vanishing state $\dfrac{AP}{A'P'} = \dfrac{AT}{\cancel{A'T'}} = \dfrac{AO}{A'O} = \dfrac{r}{r'}$; $\left(\dfrac{AT}{A'T'}\right.$

$\therefore \dfrac{\text{circumference A B C}}{\text{circumference A}'\text{B}'\text{C}'} = \dfrac{r}{r'}$; or the circumferences of circles are to one another as the radii.

2. Cor. 1. From this proposition it appears, that in the circle,

$$\frac{\text{circumference}}{\text{diameter}} = \text{a constant number.}$$

This number cannot be expressed in finite terms, but an approximation to it of any degree of accuracy may be made. (Peacock's Examples on the Diff. Cal. Chap. VIII.) It is always called π, and to twelve places of decimals

$$\pi = 3.141592653590.$$

3. Cor. 2. Since $\frac{\text{circumference}}{2r} = \pi$,

$$\therefore \text{circumference} = 2\pi r.$$

4. Cor. 3. When $r = 1$,

$$\text{circumference} = 2\pi.$$

5. Lemma. *The area of a circle* $= \frac{1}{2}$ *(radius). (circumference.*

For $\frac{\text{area ABC}}{\text{circumference ABC}} = \frac{\text{Sector AOP}}{\text{arc AP}}$, (Euc. VI. 33.)

$$\therefore \frac{\text{area ABC}}{\frac{1}{2} r.\ (\text{circumference ABC})} = \frac{\text{sector AOP}}{\frac{1}{2} r.\ (\text{arc AP})};$$

and this ratio will be true, whatever be the magnitude of the angle at O: let it be diminished without limit; then AP coincides with AT, and the sector AOP becomes = triangle AOT $= \frac{1}{2}$. AO. AT $= \frac{1}{2} r$. (arc AP);

$\therefore$ area of the circle ABC $= \frac{1}{2} r$. (circumference ABC.)

6. Cor. 1. The area of a circle $= \frac{1}{2} r$. (circumference.)

$$= \tfrac{1}{2} r.\ 2\pi r,$$
$$= \pi r^2.$$

7. Cor. 2. When radius = 1, area = π.

ON THE DIVISION OF AN ANGLE INTO DEGREES, MINUTES, SECONDS, &c.

8. (I.) *The sexagesimal division.*

In this division a right angle is divided with 90 equal angles, each of which is called a *degree* and is taken for the *angular unit*. A degree is again divided into 60 equal angles, each of these is called a *minute*. A minute is thus subdivided in 60 *seconds*; a second into 60 thirds, and so on. Degrees, minutes and seconds are thus denoted.

$$23^\circ \quad 27' \quad 34'' \quad 14''' \quad 7^{iv}$$

That is, 23 degrees, 27 minutes, 34 seconds, 14 thirds, and 7 fourths. Thirds and fourths, however, are seldom found, as mathematicians find it more convenient to use in their place their value expressed in decimal parts of a second.

By this division four right angles, or the whole angular space about a point, is divided into 360 degrees.

9. (II). *The decimal division.*

An attempt has been made by some foreign mathematicians to supersede this by another division of angles more convenient for the established decimal scale of numeration. They have divided a right angle into 100 equal parts, or angular units, and each of them has been called a *grade;* each grade has been divided into 100 minutes, each minute into 100 seconds, and so on: thus,

32 grades, 14 minutes, 6 seconds, are written,

$$32^g \quad 14^{`} \quad 6^{``}.$$

Hence all arithmetic operations would be performed with much less labour, than in the old division. But the apparent advantages of the alteration have not been sufficient to induce many to undergo the inconvenience of so great a deviation

from the common division; and, on this account, the attempt will most probably prove entirely unsuccessful.

10. Since, however, several scientific works were printed according to this new scale, formulæ are necessary for converting degrees, minutes, &c. of one division into the corresponding degrees, minutes, &c. of the other.

Let E = the number of *degrees*, &c. in any angle

and F = ... *grades*, &c. in the same angle.

Then, since a right angle contains 90 English, and 100 foreign degrees,

$$\frac{E^\circ}{F^g} = \frac{90}{100} = \frac{9}{10},$$

$$\frac{E'}{F'} = \frac{90}{100} \cdot \frac{60}{100} = \frac{9}{10} \cdot \frac{6}{10},$$

$$\frac{E''}{F''} = \frac{9}{10} \cdot \left(\frac{6}{10}\right)^2,$$

$$\frac{E'''}{F'''} = \frac{9}{10} \cdot \left(\frac{6}{10}\right)^3,$$

&c. = &c.

11. COR. $E^\circ = \dfrac{9\,F^g}{10} = F^g - \dfrac{F^g}{10}$, and $F^g = \dfrac{10\,E^\circ}{9}$.

12. Ex. 1. Convert $67^\circ\ 10'\ 26''\ 47'''\ 45^{iv}\ 36^{v}$ into degrees, minutes, &c. of the *decimal* notation.

First, reduce the minutes, &c. to the decimal of a degree, in the following manner:

$$\begin{array}{r}
60\overline{)36^{v}} \\
60\overline{)45^{iv}.\ 6} \\
60\overline{)47'''.\ 76} \\
60\overline{)26''.\ 796} \\
60\overline{)10'.\ 4466} \\
67^\circ.\ 17411 = 67^\circ\ 10'\ 26''\ 47'''\ 45^{iv}\ 36^{v}.
\end{array}$$

$$\text{Hence } \frac{10\ \mathrm{E}^\circ}{9} = \frac{10}{9} \times 67^\circ.17411 = 74^\circ.6379,$$

$$\therefore\ \mathrm{F}^g = 74^g\ 63^{`}\ 79^{``}, \text{ the answer required.}$$

13. Ex. 2. Convert $32^g\ 14^{`}\ 6^{``}$ into common degrees.

Here $\mathrm{F}^g = 32^g\ 14^{`}\ 16^{``} = 32^g.146$,

$$\therefore\ \frac{\mathrm{F}^g}{10} = 3^g.2146,$$

$$\therefore\ \mathrm{F}^g - \frac{\mathrm{F}^g}{10} = 28^g.9314$$

$\therefore\ \mathrm{E}^\circ = 28^\circ.9314$, the answer required, or if the decimal be reduced into minutes, &c. thus,

$$\begin{array}{r} 9314 \\ 60 \\ \hline 55'.884 \\ 60 \\ \hline 53''.04 \\ 60 \\ \hline 2'''.4 \\ 60 \\ \hline 24^{\mathrm{iv}} \end{array}$$

which gives $55'\ 53''\ 2'''\ 24^{\mathrm{iv}}$.

$$\therefore\ \mathrm{E}^\circ = 28^\circ\ 55'\ 53''\ 2'''\ 24^{\mathrm{iv}}.$$

14. Another emendation of the established division of angles has been proposed. The object of which is to preserve the division of a right angle into 90°, but to abolish the sexagesimal division into minutes, seconds, &c. and to express angles in degrees and decimals of a degree: or, which is the same thing, to call the two first figures after the decimal point *minutes*, the two next *seconds*, and so on. This would agree with the decimal scale of numeration; and, since the new minutes, seconds, &c. would be less than the old, in the ratio of 6 to 10, 6^2 to 10^2, &c. it would require fewer figures in the new, than in the old, to express an angle to the same degree

of accuracy. If some accurate tables were printed, with this alteration, there is every reason to suppose, that it would soon be generally adopted; for the saving of arithmetic labour would be great, and as the old *degrees* would be preserved, there would not be that abrupt change, which has proved fatal to the general adoption of the foreign division.

15. Some mathematicians, in their algebraical calculations, with a view to simplify trigonometric formulæ, use a right angle for the *angular unit*, and therefore write down as the value of any other angle the *ratio* which it bears to a right angle.

16. PROP. *If a' be an arc of a circle, of which the radius is r, subtending an angle* A *at the centre,* $A = \frac{180°}{\pi} \cdot \frac{a'}{r}$.

$$\text{For } \frac{A}{a'} = \frac{\text{four right angles}}{\text{circumference}}. \text{ (Euc. VI. 33.)}$$
$$= \frac{360°}{2\pi r}. \text{ (art. 3.)}$$
$$\therefore A = \frac{180°}{\pi} \cdot \frac{a'}{r}.$$

17. COR. 1. $A \propto \frac{a'}{r}$. Hence the fraction $\frac{\text{arc}}{\text{radius}}$ is taken as the *measure* of the angle which is subtended at the centre. If a be the value of a' when r = a linear unit $= 1$, $A = \frac{180°}{\pi} a$; and a, or the arc itself *measures* the angle; thus $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{5}$, and $\frac{\pi}{6}$ *measure* respectively, 180°, 90°, 60°, 45°, 36° and 30°.

18. COR. 2. By making $a' = r$, it follows, that the angle which is subtended by an arc equal to the radius of the circle

$$= \frac{180^\circ}{\pi}$$
$$= 57^\circ.\ 29577$$
$$= 57^\circ\ 17'\ 44''.\ 772$$
$$= 206264''.\ 772$$
$$= 63^g.\ 66197.$$

19. Cor. 3. $A = \frac{180^\circ}{\pi} \cdot \frac{a'}{r} = \frac{180^\circ}{\pi} \cdot a$

$$\therefore \frac{a'}{r} = a$$

$$\therefore a' = a\ r.$$

Def. I. The *complement* of an *arc* is that arc subtracted from a quadrant.

Def. II. The *complement* of an *angle* is that angle subtracted from a right angle.

Def. III. The *supplement* of an *arc* is that arc subtracted from a semicircle.

Def. IV. The *supplement* of an *angle* is that angle subtracted from two right angles.

20. Cor. Hence in a right angled triangle the acute angles are complements of each other; and in any triangle one of the angles is the supplement of the sum of the other two. (Euc. I. 32.)

21. *Definitions of the Trigonometric Functions of an arc.*

Def. V. The *sine* of an *arc* is the right line drawn from

one extremity of the arc perpendicularly to the diameter passing through the other extremity.

Def. VI. The *cosine* of an *arc* is the sine of the complement of that arc.

Def. VII. The *tangent* of an *arc* is the right line drawn from one extremity of the arc touching the circle, and terminated by the radius produced through the other extremity.

Def. VIII. The *cotangent* of an *arc* is the tangent of the complement of that arc.

Def. IX. The *secant* of an *arc* is the line drawn from the centre through one extremity of the arc, and terminated by the tangent to the other extremity.

Def. X. The *cosecant* of an *arc* is the secant of the complement of that arc.

Def. XI. The *versed sine* of an *arc* is the part of the diameter intercepted between one extremity of the arc, and the foot of the sine drawn from the other extremity.

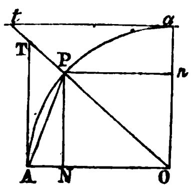
Def. XII. The *chord* of an *arc* is the line joining the extremities of an arc.

Besides these, there are two more trigonometric functions, which are useful in logarithmic calculations, viz. the *coversed sine,* and the *suversed sine* of an *arc.*

Def. XIII. The *coversed sine* of an *arc* is the versed sine of the complement of that arc.

DEF. XIV. The *suversed sine* of an *arc* is the versed sine of the supplement of that arc.

22. To illustrate these definitions, take O the centre of the circle AP a.



Let the radii OA, Oa, be at right angles to each other, and therefore APa a quadrant. From P any point in the circumference, draw PN, Pn, perpendiculars to OA, Oa; therefore (Euc. I. 34.) PN $=$ On, and Pn $=$ ON. From A and a draw AT, at, at right angles to OA, Oa, and therefore touching the circle: join OP, and produce it to meet AT, at in T and t; join also AP: then,

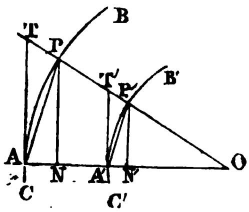
Pa is the complement of the arc AP,
PN . . . . sine,
ON $=$ Pn cosine,
AT . . . . tangent,
at . . . . cotangent,
OT . . . . secant,
Ot . . . . cosecant,
AN . . . . versed sine,
an . . . . coversed sine,
AP . . . . chord.

23. The words sine, cosine, &c. are thus abbreviated, sin, cos, &c. and the letter f is used as a general symbol for any of the trigonometric functions: thus $f\,a$ means the sine of a, or the cosine, or the tangent, or any other trigonometric function of a.

24. *If α and α' be arcs of circles to radii 1, and r subtending the same angle at the centre,*

$$f\,\alpha = \frac{1}{r} f\,\alpha'.$$

Construct the figure as in (art. 1.), join AP, A′ P′, let AO = 1, and A′O = r: let AP = α, therefore A′P′ = α'.



And by similar triangles;

$$\frac{PN}{PO} = \frac{P'N'}{P'O}, \therefore \sin\alpha = \frac{\sin\alpha'}{r},$$

$$\frac{AT}{OA} = \frac{A'T'}{A'O}, \therefore \tan\alpha = \frac{\tan\alpha'}{r},$$

$$\frac{NO}{AO} = \frac{N'O}{A'O},$$

$$\therefore 1 - \frac{NO}{AO} = 1 - \frac{N'O}{A'O},$$

Or $$\frac{AN}{AO} = \frac{A'N'}{A'O}, \therefore \text{vers}\,\alpha = \frac{\text{vers}\,\alpha'}{r},$$

$$\frac{AP}{AO} = \frac{A'P'}{A'O}, \therefore \text{chord}\,\alpha = \frac{\text{chord}\,\alpha'}{r}.$$

The same is $\therefore$ true for the complements of α and α': hence,

$$\sin(\text{comp. of } \alpha) = \frac{\sin(\text{comp. of } \alpha')}{r},$$

$\therefore$ by the definition, $\cos\alpha = \dfrac{\cos\alpha'}{r}$,

similarly $\cot\alpha = \dfrac{\cot\alpha'}{r}$,

and $\text{covers}\,\alpha = \dfrac{\text{covers}\,\alpha'}{r}$,

or, in general, $f\,a = \frac{1}{r} \cdot f\,a'$.

25. Cor. 1. In (art. 17.) $\frac{a'}{r}$ was taken as the measure of the angle at the centre of the circle, because it remained unchanged, however the radius might be altered. For the same reason $\frac{\sin a'}{r}$, $\frac{\cos a'}{r}$, &c. which by this proposition are proved to remain the same, when different values are given to the radius, are assumed as the *sine, cosine,* &c. of the *angle* AOP, or in other words.

26. Def. XV. Any trigonometric function of an *angle* is the same function of the *arc subtending that angle* at the centre of a circle, *divided by the radius of the circle.*

27. Prop. *If the angle* AOP $=$ A, *then* $f\,A = f\,a$.

For by Def. XV. $f\,A = \frac{1}{r} f\,a'$.

But by (art. 24.) $f\,a = \frac{1}{r} f\,a'$.

$\therefore f\,A = f\,a$.

28. Cor. Hence if any formula of trigonometric functions be proved for *circular arcs to radius unity,* it will be true when the corresponding *angles* are substituted for the *arcs.* Thus in the formula,

$$\sin a = \sin(\pi - a) = -\sin(\pi + a) = -\sin(2\pi - a).$$

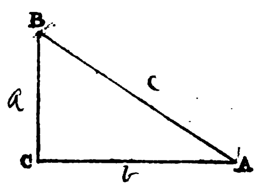
Since the angles at the centre subtended by π and 2π are $180°$ and $360°$, if A be the angle subtended by a, then, by the Prop. the formula becomes

$$\sin A = \sin(180° - A) = -\sin(180° + A) = -\sin(360 - A).$$

29. The student must remember, that this is true only for the *trigonometric functions*, and does not apply to the arcs and angles themselves. Although $f\ A = f\ a$, it must not be concluded, that the numerical values of A and a are the same, since $A = \frac{180}{\pi} a$; A having reference to an angular, and a to a linear unit.

30. PROP. *In any right angled triangle* ABC *of which the two acute angles are* A *and* B, *the right angle* C, *and the opposite sides a, b, c, then*

$$\sin A = \frac{a}{c}.$$



For, by describing a circle with centre A and radius AB ($= c$), BC ($= a$) would be the sine of the *arc* subtending A; therefore by Def. XV.

$$\sin A = \frac{a}{c}.$$

31. COR. $\sin(90° - A) = \sin B = \frac{AC}{AB}$.

$$\therefore \cos A = \frac{b}{c}.$$

32. PROP. $Tan\ A = \frac{a}{b}$,

$$cot\ A = \frac{b}{a}.$$

For, by describing a circle with centre A and radius AC

$(= b)$ BC $(= a)$ would be the tangent of the *arc* subtending A, therefore

$$\tan A = \frac{a}{b}.$$

$$\text{Also } \tan (90^\circ - A) = \tan B = \frac{AC}{BC}$$

$$\therefore \cot A = \frac{b}{a}.$$

33. Cor. $$\text{Sec } A = \frac{AB}{AC} = \frac{c}{b}.$$

$$\text{cosec } A = \sec B = \frac{AB}{BC} = \frac{c}{a}.$$

34. Prob. *Any formula consisting of trigonometric functions of an arc having been calculated to radius unity, it is required to transfer it to radius r.*

Since $\alpha = \frac{\alpha'}{r}$, $\sin \alpha = \frac{\sin \alpha'}{r}$, &c. it will only be necessary to divide every separate trigonometric function by r: thus $e^{\alpha\sqrt{-1}}(\cos \alpha + \sqrt{-1} \sin \alpha)$, becomes, to rad $= r$, $e^{\frac{\alpha'}{r}\sqrt{-1}}\left(\frac{\cos \alpha'}{r} + \sqrt{-1}\,\frac{\sin \alpha'}{r}\right)$.

35. Prop. *To transfer an equation of trigonometric functions from radius unity to radius r, it will only be necessary to multiply, or divide by some power of r, so as to reduce every term to the same dimension.*

For let $a (\sin \alpha)^m + b (\cos \beta)^n + c (\tan \gamma)^p + \ldots = 0$, where α, β, γ, &c. are *arcs* to rad $= 1$, and a, b, c, &c. are any numerical quantities; let the corresponding arcs to rad $= r$, be α', β', γ', &c.

$$\therefore a\left(\frac{\sin \alpha'}{r}\right)^m + b\left(\frac{\cos \beta'}{r}\right)^n + c\left(\frac{(\tan \gamma'}{r}\right)^p + \ldots = 0,$$

$\therefore ar^{n+p}(\sin\alpha')^m + br^{m+p}(\cos\beta')^n + cr^{m+n}(\tan\gamma')^p + \ldots = 0,$
where each term is of $m + n + p$ dimensions.

The values of the radius to which trigonometric function of arcs are most commonly calculated are,

(1.) $r = 1$. This is used for trigonometric formulæ, because it simplifies the expressions.

(2.) $r = 10^4$. This is used for tables of natural sines, &c.

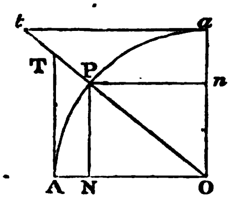
(3.) $r = 10^{10}$. This is used for tables of logarithmic sines, &c.

The uses of the two latter will be shown in a subsequent section.

Unless the contrary be expressed, the radius used in the following treatise will be unity. Angles will be denoted by the Roman capitals A, B, C, . . . and arcs to radius unity by the small Greek characters α, β, γ. . . .

36. Prop.
$$(Sin\ \alpha)^2 + (cos\ \alpha)^2 = 1,$$
$$(sec\ \alpha)^2 - (tan\ \alpha)^2 = 1,$$
$$(cosec\ \alpha)^2 - (cot\ \alpha)^2 = 1.$$

For construct the figure as in p. 10; and in the right angled triangles TAO, PNO, O*ta*, we have,



$$PN^2 + NO^2 = PO^2, \therefore (\sin\alpha)^2 + (\cos\alpha)^2 = 1;$$
$$OT^2 - AT^2 = AO^2, \therefore (\sec\alpha)^2 - (\tan\alpha)^2 = 1;$$
$$Ot^2 - at^2 = Oa^2, \therefore (\text{cosec}\,\alpha)^2 - (\cot\alpha)^2 = 1.$$

37. Prop.
$$Tan\ \alpha = \frac{sin\ \alpha}{cos\ \alpha};\ cot\ \alpha = \frac{cos\ \alpha}{sin\ \alpha};\ tan\ \alpha = \frac{1}{cot\ \alpha},$$

For, by similar triangles in the same figure,
$$\frac{AT}{AO} = \frac{PN}{NO}, \therefore \tan\ \alpha = \frac{\sin\ \alpha}{\cos\ \alpha};$$

$$\frac{at}{Oa} = \frac{NO}{PN}, \therefore \cot \alpha = \frac{\cos \alpha}{\sin \alpha};$$

$$\frac{AT}{AO} = \frac{Oa}{at}, \therefore \tan \alpha = \frac{1}{\cot \alpha};$$

38. Prop. *Sec* $\alpha = \dfrac{1}{\cos \alpha}$; *cosec* $\alpha = \dfrac{1}{\sin \alpha}$;

For $$\frac{OT}{OA} = \frac{OP}{ON}, \therefore \sec \alpha = \frac{1}{\cos \alpha};$$

$$\frac{Ot}{Oa} = \frac{OP}{PN}, \therefore \text{cosec}\, \alpha = \frac{1}{\sin \alpha}:$$

39. Cor. $AN = AO - NO, \therefore \text{versin}\, \alpha = 1 - \cos \alpha;$

$an = aO - nO, \therefore \text{coversin}\, \alpha = 1 - \sin \alpha.$

40. Prop. $f(2 i \pi \pm \alpha) = f(\pm \alpha)$.
i being any integral number.

For, *firstly*, it is evident that if to the arc AP there be added any multiple of the whole circumference, or if it be subtracted from it, the point P will still remain the termination of the arc. Therefore the two extreme points will not have changed their position; hence it follows that all the trigonometric functions which depend only on the extremities of the arc will remain unchanged; therefore

$$\sin (2 i \pi + \alpha) = \sin \alpha,$$
$$\cos (2 i \pi + \alpha) = \cos \alpha,$$
$$\&c. = \&c.$$

or, in general terms,

$$f(2 i \pi + \alpha) = f \alpha.$$

Secondly, by the principles of analytical geometry, if lines drawn in one direction be assumed positive, the lines drawn in an opposite direction must be reckoned negative. We shall assume the arc A_1P_1 positive (see fig. in art. ~~42~~.) Hence 44 the arc A_1P_4 measured from the same point in an opposite direction must be reckoned negative. The trigonometric function will evidently be the same, whether we consider $A_1A_3P_4$ the arc, or A_1P_4;

$$\therefore f(2\pi - a) = f(-a),$$

hence, by the former part of this article,

$$f(2i\pi - a) = f(2\pi - a) = f(-a).$$

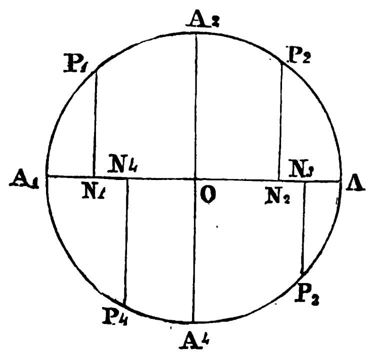
$$\therefore f(2i\pi \pm a) = f(\pm a).$$

41. Cor. 1. Having joined O_1, P_1, O_1, P_4, in the same figure, from the same considerations, if the angle A_1OP_1 be assumed positive, the angle A_1OP_4 on the opposite side of A_1O must be considered negative; and, as in the proposition,

$$f(2i\,180^\circ \pm A) = f(\pm A).$$

42. Prop. *To trace the changes of magnitude, and of algebraic sign, which the sine and cosine of an arc undergo in different parts of a circle.*

Describe the circle $A_1\,A_2\,A_3\,A_4$ with radius 1. Draw the diameters $A_1\,A_3$, $A_2\,A_4$, at right angles to each other and intersecting in O, which is therefore the centre of the circle.



Take P_1, P_2, P_3, P_4, points in the first, second, third, and fourth quadrants. Draw P_1N_1, P_2N_2, P_3N_3, P_4N_4, perpendiculars to $A_1\,A_3$.

Now the trigonometric functions of arcs terminating in the first quadrant are assumed positive; therefore they will be positive, or negative in the remaining quadrants according as they are measured in the same, or different directions from

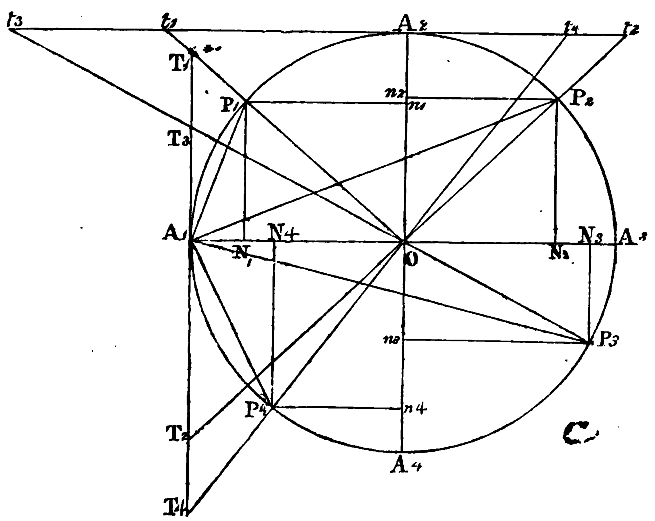
those of the first quadrant. Hence P_2N_2 the sine of the second quadrant is positive; P_3N_3 and P_4N_4 the sines of the third and fourth quadrants are negative.

For the same reason ON_2 ON_3 the cosines in the second and third quadrants are negative, and ON_4 the cosine in the fourth quadrant is positive.

In the first quadrant the sine is nothing at the beginning of the arc, and continually increases until P_1 is at A_2, and then the sine is A_2O, or 1. It then decreases in the third quadrant until it vanishes at A_3. In the third quadrant as P_3 moves from A_3 to A_4 the sine increases negatively from 0 to $-$ 1. It then decreases in the fourth quadrant until at A_1 it vanishes again.

The cosine is equal to 1, when P_1 is at A_1. It decreases through the first quadrant until at A_2 it vanishes. In the second quadrant it begins with 0, and increases negatively to $-$ 1. In the third quadrant it changes from $-$ 1 to 0, and in the fourth from 0 to 1.

43. The annexed figure and table give the changes in sign and magnitude of the remaining trigonometric functions.



	Sine.	Cosine.	Tangent.	Cotangent.	Secant.	Cosecant.	Versine.	Coversine.	Chord.
1st Quad.	P_1N_1	ON_1	A_1T_1	A_2t_1	OT_1	Ot_1	A_1N_1	A_2n_1	A_1P_1
2nd.	P_2N_2	ON_2	A_1T_2	A_2t_2	OT_2	Ot_2	A_1N_2	A_2n_2	A_1P_2
3rd.	P_3N_3	ON_3	A_1T_3	A_2t_3	OT_3	Ot_3	A_1N_3	A_2n_3	A_1P_3
4th.	P_4N_4	ON_4	A_1T_4	A_2t_4	OT_4	Ot_4	A_1N_4	A_2n_4	A_1P_4
1st Quad.	0 to 1	1 to 0	0 to ∞	∞ to 0	1 to ∞	∞ to 1	0 to 1	1 to 0	0 to $\sqrt{2}$
2nd.	1 to 0	0 to -1	$-\infty$ to 0	0 to $-\infty$	$-\infty$ to -1	1 to ∞	1 to 2	0 to 1	$\sqrt{2}$ to 2
3rd.	0 to -1	-1 to 0	0 to ∞	∞ to 0	-1 to $-\infty$	$-\infty$ to -1	2 to 1	1 to 2	2 to $\sqrt{2}$
4th.	-1 to 0	0 to 1	$-\infty$ to 0	0 to $-\infty$	∞ to 1	-1 to $-\infty$	1 to 0	2 to 1	$\sqrt{2}$ to 0

Since it has been proved that

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}, \quad \sec \alpha = \frac{1}{\cos \alpha}, \quad \text{versin } \alpha = 1 - \cos \alpha,$$

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}, \quad \text{cosec } \alpha = \frac{1}{\sin \alpha}, \quad \text{coversin } \alpha = 1 - \sin \alpha;$$

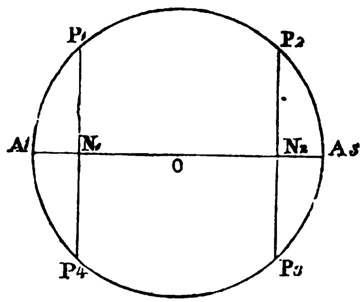
the variations in magnitude and algebraic sign of the remaining trigonometric functions may be deduced from those of the sine and cosine.

The secants of the first and third quadrants have different signs, because in the first quadrant the secant is drawn *towards* P_1 the extremity of the arc, and in the third quadrant, *from* the extremity P_3. For the same reason the secants in the second and fourth quadrants have different signs. The same observations may be made concerning the cosecants.

44. Prop.

Sin $\alpha = (\pi - \alpha) = -$ *sin* $(\pi + \alpha) = -$ *sin* $(2\pi - \alpha)$.

Let O be the centre and A_1OA_3 the diameter of a circle to radius unity.



In A_1A_3 take $ON_1 = ON_2$ on opposite sides of O. Draw $P_1N_1P_4$, $P_2N_2P_3$ at right angles to A_1A_3: these lines will therefore be bisected (Euc. III. 3.) in N_1, N_2; and, consequently, the arcs $P_1A_1P_4$, $P_2A_3P_4$, will be bisected in A_1, A_3.

But, because $ON_1 = ON_2$, from (Euc. III. 14.) $P_1P_4 = P_2P_3$, and, consequently, $P_1A_1P_4 = P_2A_3P_3$,

$$A_1P_1 = A_3P_2 = A_3P_3 = A_1P_4,$$

$$\text{and} \quad P_1N_1 = P_3N_2 = P_2N_2 = P_4N_1,$$

or $\sin A_1P_1 = \sin A_1P_2 = -\sin A_1A_3P_3 = -\sin A_1A_3P_4$,

$\therefore \sin a = \sin(\pi - a) = -\sin(\pi + a) = -\sin(2\pi - a)$.

45. COR. 1. $P_1P_4 = 2\,P_1N_1$ or chord $2\,a = 2 \sin a$.

46. PROP.

Cos $a = -\cos(\pi - a) = -\cos(\pi + a) = \cos(2\pi - a)$

For, in the same figure,

$$ON_1 = \cos A_1P_1 = \cos A_1A_3P_4,$$

$$ON_2 = -\cos A_1P_2 = -\cos A_1A_3P_3,$$

$\therefore \cos a = -\cos(\pi - a) = -\cos(\pi + a) = \cos(2\pi - a)$.

47. COR. 1. Hence, by (art. 40),

$$\sin(-a) = \sin(2\pi - a) = -\sin a,$$

$$\text{and} \quad \cos(-a) = \cos(2\pi - a) = \cos a.$$

48. COR. 2. $\tan a = \dfrac{\sin a}{\cos a}$, $\cot a = \dfrac{\cos a}{\sin a}$:

$\therefore \tan a = -\tan(\pi - a) = \tan(\pi + a) = -\tan(2\pi - a)$;

and $\cot a = -\cot(\pi - a) = \cot(\pi + a) = -\cot(2\pi - a)$;

also, $\tan(-a) = -\tan a$, $\cot(-a) = -\cot a$.

49. COR. 3. $\sec a = \dfrac{1}{\cos a}$, and $\operatorname{cosec} a = \dfrac{1}{\sin a}$:

$\therefore \sec a = -\sec(\pi - a) = -\sec(\pi + a) = \sec(2\pi - a)$;

$\operatorname{cosec} a = \operatorname{cosec}(\pi - a) = -\operatorname{cosec}(\pi + a) = -\operatorname{cosec}(2\pi - a)$;

$$\text{hence } \sec(-a) = \sec a,$$

$$\text{and } \operatorname{cosec}(-a) = -\operatorname{cosec} a.$$

50. Prop.

$$\textit{Sin}\, a = -\textit{cos}\left(\frac{\pi}{2} + a\right) = -\textit{cos}\left(\frac{3\pi}{2} - a\right) = \textit{cos}\left(\frac{3\pi}{2} + a\right).$$

For $\cos a = -\cos(\pi - a) = -\cos(\pi + a) = \cos(2\pi - a)$.

For a substitute $\frac{\pi}{2} - a$,

$$\therefore \cos\left(\frac{\pi}{2} - a\right) = -\cos\left(\pi \quad \left(\frac{\pi}{2} - a\right)\right) =$$

$$-\cos\left(\pi + \left(\frac{\pi}{2} - a\right)\right) = \cos\left(2\pi - \left(\frac{\pi}{2} - a\right)\right),$$

$$\therefore \sin a = -\cos\left(\frac{\pi}{2} + a\right) = -\cos\left(\frac{3\pi}{2} - a\right) = \cos\left(\frac{3\pi}{2} + a\right).$$

51. Cor. 1. Similarly,

$$\cos a = \sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right) = -\sin\left(\frac{3\pi}{2} - a\right)$$
$$= -\sin\left(\frac{3\pi}{2} + a\right).$$

$$\tan a = \cot\left(\frac{\pi}{2} - a\right) = -\cot\left(\frac{\pi}{2} + a\right) = \cot\left(\frac{3\pi}{2} - a\right)$$
$$= -\cot\left(\frac{3\pi}{2} + a\right).$$

$$\cot a = \tan\left(\frac{\pi}{2} - a\right) = -\tan\left(\frac{\pi}{2} + a\right) = \tan\left(\frac{3\pi}{2} - a\right)$$
$$= -\tan\left(\frac{3\pi}{2} + a\right).$$

$$\sec a = \operatorname{cosec}\left(\frac{\pi}{2} - a\right) = \operatorname{cosec}\left(\frac{\pi}{2} + a\right) = -\operatorname{cosec}\left(\frac{3\pi}{2} - a\right)$$
$$= -\operatorname{cosec}\left(\frac{3\pi}{2} + a\right).$$

$$\operatorname{cosec} a = \sec\left(\frac{\pi}{2} - a\right) = -\sec\left(\frac{\pi}{2} + a\right) = -\sec\left(\frac{3\pi}{2} - a\right)$$
$$= \sec\left(\frac{3\pi}{2} + a\right).$$

52. PROP.

Sin a = *sin* $(2\,i\,\pi + a)$ = *sin* $\{(2\,i + 1)\,\pi - a\}$ $= -$ *sin* $\{(2i + 1)\,\pi + a\}$ $= -$ *sin* $\{(2\,i + 2)\,\pi - a\}$; *where any whole number may be substituted for i in each of the expressions.*

For $\sin a = \sin(\pi - a) = -\sin(\pi + a) = -\sin(2\pi - a)$: but $\sin a = \sin(2\,i\,\pi + a)$: hence by substituting $(2\,i\,\pi + a)$ for a;

$$\sin a = \sin(2\,i\,\pi + a) = \sin(2\,i\,\pi + \pi - a)$$
$$= -\sin(2\,i\,\pi + \pi + a) = -\sin(2\,i\,\pi + 2\,\pi - a)$$
$$= \sin\{(2\,i + 1)\,\pi - a\} = -\sin\{(2\,i + 1)\,\pi + a\}$$
$$= -\sin\{(2\,i + 2)\,\pi - a\}.$$

53. COR. 1. $\sin a = \cos\left(\frac{\pi}{2} - a\right) = -\cos\left(\frac{\pi}{2} + a\right) = -\cos\left(\frac{3\pi}{2} - a\right) = \cos\left(\frac{3\pi}{2} + a\right)$; but $\cos a = \cos(2\,i\,\pi + a)$,

$$\therefore \sin a = \cos\left((4\,i + 1)\frac{\pi}{2} - a\right) = -\cos\left((4\,i + 1)\frac{\pi}{2} + a\right)$$
$$= -\cos\left((4\,i + 3)\frac{\pi}{2} - a\right) = \cos\left((4\,i + 3)\frac{\pi}{2} + a\right).$$

54. COR. 2. Similarly,

$$\cos a = \cos(2\,i\,\pi + a) = -\cos\{(2\,i + 1)\,\pi - a\}$$
$$= -\cos\{(2\,i + 1)\,\pi + a\} = \cos\{(2\,i + 2)\,\pi - a\}:$$

and $\cos a = \sin\left((4\,i + 1)\frac{\pi}{2} - a\right) = \sin\left((4\,i + 1)\frac{\pi}{2} + a\right)$

$$= -\sin\left((4\,i + 3)\frac{\pi}{2} - a\right) = -\sin\left((4\,i + 3)\frac{\pi}{2} + a\right).$$

55. COR. 3. By making $a = 0$;

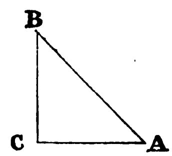
$$\sin(2\,i\,\pi) = \sin(2\,i + 1)\,\pi = 0.$$
$$\sin(4\,i + 1)\frac{\pi}{2} = -\sin(4\,i + 3)\frac{\pi}{2} = 1.$$

$$\cos(2\,i\,\pi) = -\cos(2\,i+1)\,\pi = 1.$$

$$\cos(4\,i+1)\,\frac{\pi}{2} = \cos\,(4i+3)\,\frac{\pi}{2} = 0.$$

56. Cor. 4. In the same manner the formulæ for the remaining trigonometric functions are capable of similar extensions.

57. Prop. *Sin* $45° = \cos 45° = \frac{1}{\sqrt{2}}$.



For, let ABC be an isosceles right angled triangle having C the right angle; therefore the sum of the angles A and B is equal to a right angle; but because CA is equal to CB, A and B are equal angles, therefore each of them half a right angle, or 45°.

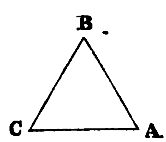
Now $AB^2 = BC^2 + AC^2 = 2\,BC^2$,

$$\therefore \frac{BC}{AB} = \frac{1}{\sqrt{2}}.$$

$$\therefore \sin\,45° = \frac{1}{\sqrt{2}}$$

Also, $\cos 45° = \frac{AC}{AB} = \frac{1}{\sqrt{2}}$.

58. Prop. *Sin* $30° = \frac{1}{2}$.



For, let ABC be an equilateral triangle; A, B, C, are

therefore equal angles, and the sum of them is two right angles or 180°, therefore each of them is 60°. But if a circle be described with centre A, and radius AB, it will pass through C.

$$\therefore \text{ chord } A = \frac{BC}{AB}, \text{ by def. XV.}$$

$$\therefore \text{ chord } 60° = 1,$$

$$\therefore 2 \sin 30° = 1, \text{ (articles 45 and 28.)}$$

$$\therefore \quad \sin 30° = \tfrac{1}{2}.$$

59. Cor. 1. $\text{Cos}\, 30° = \sqrt{\{1-(\sin 30°)^2\}} = \sqrt{(1-\tfrac{1}{4})} = \frac{\sqrt{3}}{2}.$

60. Cor. 2. $\text{Sin}\, 60° = \cos (90° - 30°) = \cos 30° = \frac{\sqrt{3}}{2}.$

61. Cor. 3. $\text{Cos}\, 60° = \sin (90° - 30°) = \sin 30° = \tfrac{1}{2}.$

62. Cor. 4. Hence the following results :—

$$\begin{aligned}
\sin 45° &= \cos 45° &&= \frac{1}{\sqrt{2}}. \\
\tan 45° &= \cot 45° &&= 1. \\
\sec 45° &= \text{cosec}\, 45° &&= \sqrt{2}. \\
\sin 30° &= \cos 60° &&= \tfrac{1}{2}. \\
\cos 30° &= \sin 60° &&= \frac{\sqrt{3}}{2}. \\
\tan 30° &= \cot 60° &&= \frac{1}{\sqrt{3}}. \\
\cot 30° &= \tan 60° &&= \sqrt{3} \\
\sec 30° &= \text{cosec}\, 60° &&= \frac{2}{\sqrt{3}}. \\
\text{cosec}\, 30° &= \sec 60° &&= 2.
\end{aligned}$$

63. The results in the annexed table follow immediately

from the theorems in articles 36, 37, 38 and 39. The values of tan α can be derived by dividing the corresponding values of sin α and cos α. Also, since every trigonometric function is known in terms of sin α and cos α; any trigonometric function may, by this table, be expressed in terms of any other.

Values of sin α.	*Values of cos α.*	*Values of tan α.*
1. $\frac{\cos\alpha}{\cot\alpha}$.	1. $\frac{\sin\alpha}{\tan\alpha}$.	1. $\frac{\sec\alpha}{\operatorname{cosec}\alpha}$.
2. $\sqrt{1-(\cos\alpha)^2}$.	2. $\sqrt{1-(\sin\alpha)^2}$.	2. $\frac{\sin\alpha}{\sqrt{1-(\sin\alpha)^2}}$.
3. $\frac{\tan\alpha}{\sqrt{1+(\tan\alpha)^2}}$.	3. $\frac{1}{\sqrt{1+(\tan\alpha)^2}}$.	3. $\frac{\sqrt{1-(\cos a)^2}}{\cos\alpha}$.
4. $\frac{1}{\sqrt{1+(\cot\alpha)^2}}$.	4. $\frac{\cot\alpha}{\sqrt{1+(\cot\alpha)^2}}$.	4. $\frac{1}{\cot\alpha}$.
5. $\frac{\sqrt{(\sec\alpha)^2-1}}{\sec\alpha}$.	5. $\frac{1}{\sec\alpha}$.	5. $\sqrt{(\sec\alpha)^2-1}$.
6. $\frac{1}{\operatorname{cosec}\alpha}$.	6. $\frac{\sqrt{(\operatorname{cosec})^2-1}}{\operatorname{cosec}\alpha}$.	6. $\frac{1}{\sqrt{(\operatorname{cosec}\alpha)^2-1}}$.
7. $\sqrt{2\operatorname{vers}\alpha-(\operatorname{vers}\alpha)^2}$.	7. $1-\operatorname{vers}\alpha$.	7. $\frac{\sqrt{2\operatorname{vers}\alpha-(\operatorname{vers}\alpha)^2}}{1-\operatorname{vers}\alpha}$.

TABLE

For reducing French Degrees, &c. to English.

1^g	0° 54′	10^g	9°
2^g	1° 48′	20^g	18°
3^g	2° 42′	30^g	27°
4^g	3° 36′	40^g	36°
5^g	4° 30′	50^g	45°
6^g	5° 24′	60^g	54°
7^g	6° 18′	70^g	63°
8^g	7° 12′	80^g	72°
9^g	8° 6′	90^g	81°
10^g	9°	100^g	90°
1`	0′ 32″.4	10`	5′ 24″
2`	1′ 4″.8	20`	10′ 48″
3`	1′ 37″.2	30`	16′ 12″
4`	2′ 9″.6	40`	21′ 36″
5`	2′ 42″	50`	27′
6`	3′ 14″.4	60`	32′ 24″
7`	3′ 46″.8	70`	37′ 48″
8`	4′ 19″.2	80`	43′ 12″
9`	4′ 51″.6	90`	48′ 36″
10`	5′ 24″	100`	54′
1``	0″.324	10``	3″.24
2``	0″.648	20``	6″.48
3``	0″.972	30``	9″.72
4``	1″.296	40``	12″.96
5``	1″.62	50``	16″.2
6``	1″.944	60``	19″.44
7``	2″.268	70``	22″.68
8``	2″.592	80``	25″.92
9``	2″.916	90``	29″.16
10``	3″.24	100``	32″.4

Example to the preceding Table.

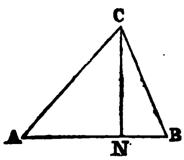
Reduce $63^g.66197$ to degrees of the ~~decimal~~ notation. sexagesimal

	60^g	=	$54°$
	3^g	=	$2°\ 42'$
	$60^{`}$	=	$32'\ 24''$
	$6^{`}$	=	$3'\ 14''.4$
	$10^{``}$	=	$3''.24$
	$9^{``}$	=	$2''.916$
	$70^{```}$	=	$.226$
$\because$	$60^g.\ 66197$	=	$57°\ 18'\ 44''.782$

SECTION II.

ON THE TRIGONOMETRIC FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ARCS.

64. Prop. *Sin* $(\alpha + \beta) = \sin\alpha . \cos\beta + \cos\alpha . \sin\beta$.



At the points A, B, in the straight line AB draw two other straight lines, making angles with AB equal to A degrees and B degrees respectively. Let the two lines be produced to meet in C. From C draw CN perpendicularly to AB. Let the angle ACB = C.

$$\therefore C = 180^\circ - (A + B).$$

$$\therefore \sin C = \sin (A + B).$$

But since the angles at N are right angles $\therefore$ by (art. 30.)

$$\sin A = \frac{CN}{CA}, \ \cos A = \frac{NA}{CA}.$$

$$\sin B = \frac{CN}{CB}, \ \cos B = \frac{NB}{CB}.$$

$$\therefore \frac{\sin B}{\sin A} = \frac{CA}{CB}.$$

Similarly, $\dfrac{\sin C}{\sin A} = \dfrac{AB}{CB}$,

$$\therefore \sin C = \sin A . \frac{AB}{CB},$$

$$= \frac{CN}{CA} \cdot \frac{NB + NA}{CB},$$
$$= \frac{CN}{CA} \cdot \frac{NB}{CB} + \frac{NA}{CA} \cdot \frac{CN}{CB}.$$
$$\therefore \sin (A + B) = \sin A. \cos B + \cos A. \sin B.$$

Let α and β be the arcs to radius unity corresponding to the angles A and B, and therefore by (art. 28)

$$\sin (\alpha + \beta) = \sin \alpha. \cos \beta + \cos \alpha. \sin \beta.$$

The figure applies to the case in which A and B are positive and acute angles. If one of the angles be negative, it must be measured on the opposite side of AB. And whatever be the values of A and B, the angles at the base of the triangle will be equal either to A and B, or to their defects from 180°, or from some multiple of 180°, and by paying attention to the algebraic signs the proof applies to angles or arcs of any magnitude whatsoever.

65. Cor. For β substitute $-\beta$.

$\therefore \sin (\alpha - \beta) = \sin \alpha. \cos (-\beta) + \cos \alpha. \sin (-\beta)$;

But $\cos (-\beta) = \cos \beta$, and $\sin (-\beta) = -\sin \beta$, (art. 47.)

$\therefore \sin (\alpha - \beta) = \sin \alpha. \cos \beta - \cos \alpha \sin \beta.$

66. Prop. *Cos* $(\alpha + \beta) = \cos \alpha. \cos \beta - \sin \alpha. \sin \beta.$

For $\sin (\alpha + \beta) = \sin \alpha. \cos \beta + \cos \alpha. \sin \beta.$

Substitute $\frac{\pi}{2} + \alpha$ for α.

$$\therefore \sin \left(\left(\frac{\pi}{2} + \alpha\right) + \beta \right) = \sin \left(\frac{\pi}{2} + \alpha\right). \cos \beta + \cos \left(\frac{\pi}{2} + \alpha\right). \sin \beta.$$

$\therefore$ by articles (50) and (51.)

$$\cos (\alpha + \beta) = \cos \alpha. \cos \beta - \sin \alpha. \sin \beta.$$

67. Cor. $\cos (\alpha - \beta) = \cos \alpha. \cos (-\beta) - \sin \alpha. \sin (-\beta)$
$= \cos \alpha. \cos \beta + \sin \alpha. \sin \beta.$

68. Prop. *Tan* $(\alpha + \beta) = \dfrac{\tan \alpha + \tan \beta}{1 - \tan \alpha. \tan \beta}.$

For $\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha.\ \cos\beta + \cos\alpha.\ \sin\beta}{\cos\alpha.\ \cos\beta - \sin\alpha.\ \sin\beta}$.

$$= \frac{\dfrac{\sin\alpha.\ \cos\beta + \cos\alpha.\ \sin\beta}{\cos\alpha.\ \cos\beta}}{\dfrac{\cos\alpha.\ \cos\beta - \sin\alpha.\ \sin\beta.}{\cos\alpha.\ \cos\beta.}}$$

$$= \frac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{1 - \dfrac{\sin\alpha}{\cos\alpha}.\ \dfrac{\sin\beta}{\cos\beta}.}$$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha.\ \tan\beta}. \quad \text{(art. 37.)}$$

69. Cor. $\operatorname{Tan}(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha.\ \tan(-\beta)}$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha.\ \tan\beta}. \quad \text{(art. 48.)}$$

70. Prop. $\left.\begin{aligned} Sin\ 2\alpha &= 2\sin\alpha.\ \cos\alpha \\ \cos 2\alpha &= (\cos\alpha)^2 - (\sin\alpha)^2 \end{aligned}\right\}$

For by making $\alpha = \beta$ in the formulæ

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha.\ \ \cos\beta + \cos\alpha.\ \ \sin\beta. \\ \cos(\alpha+\beta) &= \cos\alpha.\ \ \cos\beta - \sin\alpha.\ \ \sin\beta. \\ \sin 2\alpha &= 2\sin\alpha.\ \ \cos\alpha. \\ \cos 2\alpha &= (\cos\alpha)^2 - (\sin\alpha)^2. \end{aligned}$$

71. Prop. $\left.\begin{aligned} Cos\ 2\alpha &= 2(\cos\alpha)^2 - 1 \\ &= 1 - 2(\sin\alpha)^2 \end{aligned}\right\}$

For $(\cos\alpha)^2 - (\sin\alpha)^2 = (\cos\alpha)^2 - (1 - (\cos\alpha)^2)$

$$\therefore \cos 2\alpha = 2(\cos\alpha)^2 - 1.$$

Also $(\cos\alpha)^2 - (\sin\alpha)^2 = 1 - (\sin\alpha)^2 - (\sin\alpha)^2$,

$$\therefore \cos 2\alpha = 1 - 2(\sin\alpha)^2.$$

72. PROP. $\left.\begin{aligned} Cos\ \alpha &= \sqrt{\ }\{\tfrac{1}{2}(1 + cos\ 2\ \alpha)\} \\ sin\ \alpha &= \sqrt{\ }\{\tfrac{1}{2}(1 - cos\ 2\ \alpha)\} \end{aligned}\right\}$

For $2\ (\cos\alpha)^2 - 1 = \cos 2\ \alpha.$

$\therefore\ 2\ (\cos\alpha)^2 = 1 + \cos 2\alpha.$

$\therefore\quad (\cos\alpha)^2 = \tfrac{1}{2}\ (1 + \cos 2\ \alpha).$

$\therefore\quad \cos\ \alpha\ = \sqrt{\ }\{\tfrac{1}{2}\ (1 + \cos 2\ \alpha)\}\ .$

Again, $1-2\ (\sin\alpha)^2\ = \cos 2\alpha.$

$\therefore\ 2\ (\sin\alpha)^2 = 1 - \cos 2\ \alpha.$

$\therefore\quad (\sin\alpha)^2 = \tfrac{1}{2}\ (1 - \cos 2\ \alpha).$

$\therefore\quad \sin\ \alpha\ = \sqrt{\ }\{\tfrac{1}{2}\ (1 - \cos 2\ \alpha)\}\ .$

73. COR. For α substitute $\frac{\pi}{4} - \alpha$, and therefore

for $2\ \alpha\ \ldots\ldots\ \frac{\pi}{2} - 2\ \alpha.$

and, since $\cos\left(\frac{\pi}{2} - 2\alpha\right) = \sin 2\ \alpha.$

$$\left.\begin{aligned} \therefore\ \cos\left(\frac{\pi}{4} - \alpha\right) &= \sqrt{\ }\tfrac{1}{2}\ (1 + \sin 2\ \alpha) \\ \text{and}\ \sin\left(\frac{\pi}{4} - \alpha\right) &= \sqrt{\ }\tfrac{1}{2}\ (1 - \sin 2\ \alpha) \end{aligned}\right\}.$$

74. PROP.

$$\left.\begin{aligned} 2\ Cos\ \alpha &= \sqrt{\ }\{(2 + \sqrt{\ }(2 + \sqrt{\ }(2 + \ldots + \sqrt{\ }(2 + 2\ cos\ 2^n\ \alpha)\} \\ 2\ sin\ \alpha &= \sqrt{\ }\{(2 - \sqrt{\ }(2 + \sqrt{\ }(2 + \ldots + \sqrt{\ }(2 + 2\ cos\ 2^n\ \alpha)\} \end{aligned}\right\}$$

Firstly, $\cos\alpha = \sqrt{\ }\tfrac{1}{2}\ (1 + \cos 2\ \alpha)$

$\therefore\ 2\cos\alpha = \sqrt{\ }(2 + 2\cos 2\ \alpha)$

$\therefore\ 2\cos 2\alpha = \sqrt{\ }(2 + 2\cos 2^2\ \alpha)$

$\therefore\ 2\cos\alpha = \sqrt{\ }\{(2 + \sqrt{\ }(2 + 2\cos 2^2\ \alpha)\}.$

Similarly,

$2\cos\alpha = \sqrt{\ }\{(2 + \sqrt{\ }(2 + \sqrt{\ }(2 + 2\cos 2^3\ \alpha)\}.$

$\therefore$ by continual substitution

$2\cos\alpha = \sqrt{\ }\{(2 + \sqrt{\ }(2 + \sqrt{\ }(2 + \ldots + \sqrt{\ }(2 + 2\cos 2^n\ \alpha)\}.$

the symbol $\sqrt{\ }$ being continued n times.

Secondly, $\sin a = \sqrt{\tfrac{1}{2}(1 - \cos 2a)}$

$\therefore\ 2\sin a = \sqrt{(2 - 2\cos 2a)}$

$= \sqrt{\{(2 - \sqrt{(2 + \sqrt{(2 + \ldots + \sqrt{(2 + 2\cos 2^n a)\}}}}}$

the symbol $\sqrt{}$ being continued n times.

75. COR. 1. In these formulæ let a be changed into $\frac{a}{2^n}$, and therefore $2^n a$ into a.

$$\therefore 2\cos\frac{a}{2^n} = \sqrt{\{(2+\sqrt{(2+\sqrt{(2+\ldots+\sqrt{(2+2\cos a)\}}}}}.$$

$$2\sin\frac{a}{2^n} = \sqrt{\{(2-\sqrt{(2+\sqrt{(2+\ldots+\sqrt{(2+2\cos a)\}}}}}.$$

76. COR. 2.

$$\operatorname{Tan}\frac{a}{2^n} = \sqrt{\frac{\{(2-\sqrt{(2+\sqrt{(2+\ldots+\sqrt{(2+2\cos a)\}}}}}{\{(2+\sqrt{(2+\sqrt{(2+\ldots+\sqrt{(2+2\cos a)\}}}}}}.^*$$

---

* Archimedes discovered that $\pi = \frac{22}{7}$ nearly, by a method virtually the same as the following:—

In corollaries 1 and 2 let $a = \frac{\pi}{3}$, and $n = 5$

$$\therefore 2\cos a = 1, \text{ and } \frac{a}{2} = \frac{\pi}{96}$$

$$\therefore \sin\frac{\pi}{96} = \tfrac{1}{2}\sqrt{\{(2 - \sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \sqrt{3})\}}}}}}$$

$$\tan\frac{\pi}{96} = \sqrt{\frac{\{(2 - \sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \sqrt{3})\}}}}}}{\{(2 + \sqrt{(2 + \sqrt{(2 + \sqrt{(2 + \sqrt{3})\}}}}}}}.$$

But, if the arc be less than a quadrant, it is greater than the sine, and less than the tangent. Therefore, $\frac{\pi}{96}$ lies between *sin* $\frac{\pi}{96}$ and *tan* $\frac{\pi}{96}$. Hence, if the arithmetic operations be performed, the result, as far as the decimal places agree in both, may be taken for an approximate value of $\frac{\pi}{96}$; and, therefore, by multiplying by 96, π may be found.

77. PROP. $\quad Tan\ a = \sqrt{\left(\frac{1 - cos\ 2\ a}{1 + cos\ 2\ a}\right)},$

For, $\sin a = \sqrt{\tfrac{1}{2}(1 - \cos 2\ a)}$
and $\cos a = \sqrt{\tfrac{1}{2}(1 + \cos 2\ a)}$
$\therefore$ by division

$$\tan a = \sqrt{\left(\frac{1 - \cos 2\ a}{1 + \cos 2\ a}\right)}.$$

78. COR. Also, since

$$\sin\left(\frac{\pi}{4} - a\right) = \sqrt{\tfrac{1}{2}(1 - \sin 2\ a)}$$

$$\cos\left(\frac{\pi}{4} - a\right) = \sqrt{\tfrac{1}{2}(1 + \sin 2\ a)}$$

$$\therefore \tan\left(\frac{\pi}{4} - a\right) = \sqrt{\left(\frac{1 - \sin 2\ a}{1 + \sin 2\ a}\right)}.$$

79. PROP. $\quad Cos\ 2\ a = \dfrac{1 - (tan\ a)^2}{1 + (tan\ a)^2}$

$$\begin{aligned}\text{For } \cos 2\ a &= 2(\cos a)^2 - 1\\ &= \frac{2}{(\sec a)^2} - 1, \text{ (art. 38.)}\\ &= \frac{2}{1 + (\tan a)^2} - 1, \text{ (art. 36.)}\\ &= \frac{1 - (\tan a)^2}{1 + (\tan a)^2}.\end{aligned}$$

80. COR. For a substitute $\dfrac{\pi}{4} - a,$

and since $\cos\left(\dfrac{\pi}{2} - 2\,a\right) = \sin 2\ a,$

Any arc whose cosine is known may be taken, and any integral value of n and π may be found in the same manner. The accuracy of the result will depend on the magnitude of n.

$$\therefore \sin 2\,\alpha = \frac{1 - (\tan(\frac{\pi}{4} - \alpha))^2}{1 + (\tan(\frac{\pi}{4} - \alpha))^2}.$$

81. **Prop.** *If* α *be an arc less than* $\frac{\pi}{4}$

$$\left.\begin{aligned} 2\sin\alpha &= \sqrt{(1 + \sin 2\,\alpha)} - \sqrt{(1 - \sin 2\,\alpha)}, \\ 2\cos\alpha &= \sqrt{(1 + \sin 2\,\alpha)} + \sqrt{(1 - \sin 2\,\alpha)}. \end{aligned}\right\}$$

For, $(\sin\alpha)^2 + (\cos\alpha)^2 = 1$

and $2\sin\alpha.\cos\alpha = \sin 2\,\alpha$.

$$\therefore (\sin\alpha)^2 \pm 2\sin\alpha.\cos\alpha + (\cos\alpha)^2 = 1 \pm \sin 2\,\alpha,$$

$$\therefore \sin\alpha + \cos\alpha = \pm\sqrt{(1 + \sin 2\,\alpha)}$$

$$\sin\alpha - \cos\alpha = \pm\sqrt{(1 - \sin 2\,\alpha)}.$$

But, since α is less than $\frac{\pi}{4}$, $\sin\alpha$ is less than $\cos\alpha$, and $\cos\alpha$ is positive.

$$\therefore \sin\alpha + \cos\alpha = \sqrt{(1 + \sin 2\,\alpha)},$$

$$\sin\alpha - \cos\alpha = -\sqrt{(1 - \sin 2\,\alpha)},$$

$\therefore$ by addition and subtraction

$$2\sin\alpha = \sqrt{(1 + \sin 2\,\alpha)} - \sqrt{(1 - \sin 2\,\alpha)},$$

$$2\cos\alpha = \sqrt{(1 + \sin 2\,\alpha)} + \sqrt{(1 - \sin 2\,\alpha)}.$$

82. Cor. 1. These expressions, which are called the *formulæ of verification,* cannot be applied to arcs greater than $\frac{\pi}{4}$ without some alteration of the algebraic signs.

Now, since

$\sin\alpha = \sin(\pi - \alpha) = -\sin(\pi + \alpha) = -\sin(2\,\pi - \alpha)$ (art.44.)

and

$\cos\alpha = -\cos(\pi - \alpha) = -\cos(\pi + \alpha) = \cos(2\,\pi - \alpha)$ (art.46.)

Also, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$. (art. 57 and 28.)

Therefore, when the arc lies between $\pi - \frac{\pi}{4}$ and $2\pi - \frac{\pi}{4}$

$$\sin \alpha + \cos \alpha = - \sqrt{(1 + \cos 2\alpha)}$$

otherwise, $\sin \alpha + \cos \alpha = + \sqrt{(1 + \sin 2\alpha)}$

when the arc lies between $\frac{\pi}{4}$ and $\pi + \frac{\pi}{4}$

$$\sin \alpha - \cos \alpha = + \sqrt{(1 - \sin 2\alpha)}$$

otherwise, $\sin \alpha - \cos \alpha = - \sqrt{(1 - \sin 2\alpha)}$.

83. PROP. *Tan* α = *cosec* 2α — *cot* 2α.

For, $\frac{\sin \alpha}{\cos \alpha} = \frac{2(\sin \alpha)^2}{2 \sin \alpha . \cos \alpha}$

$$= \frac{1 - \cos 2\alpha}{\sin 2\alpha} \text{ (art. 71.)}$$

$$= \frac{1}{\sin 2\alpha} - \frac{\cos 2\alpha}{\sin 2\alpha}$$

$\therefore \tan \alpha = \operatorname{cosec} 2\alpha - \cot 2\alpha$.

84. PROP. *Cot* α = *cosec* 2α + *cot* 2α.

For, $\frac{\cos \alpha}{\sin \alpha} = \frac{2(\cos \alpha)^2}{2 \sin \alpha . \cos \alpha}$

$$= \frac{1 + \cos 2\alpha}{\sin 2\alpha} \text{ (art. 71.)}$$

$$= \frac{1}{\sin 2\alpha} + \frac{\cos 2\alpha}{\sin 2\alpha}$$

$\therefore \cot \alpha = \operatorname{cosec} 2\alpha + \cot 2\alpha$.

85. PROP. *Cosec* $2\alpha = \frac{1}{2}$ (*tan* α + *cot* α).

For $\frac{1}{\sin 2\alpha} = \frac{(\sin \alpha)^2 + (\cos \alpha)^2}{2 \sin \alpha \cos \alpha}$ (art. 70.)

$$= \frac{1}{2}\left(\frac{\sin \alpha}{\cos \alpha} + \frac{\cos \alpha}{\sin \alpha}\right)$$

$\therefore \operatorname{cosec} 2\alpha = \frac{1}{2}(\tan \alpha + \cot \alpha)$.

86. PROP. $Tan\ 2\,\alpha = \dfrac{2\ tan\ \alpha}{1 - (tan\ \alpha)^2}.$

For, $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha,\ \tan\beta}.$

Let $\alpha = \beta$,

$$\therefore\ \tan 2\,\alpha = \frac{2 \tan\alpha}{1 - (\tan\alpha)^2}.$$

87. COR. 1. $\dfrac{1}{\tan 2\,\alpha} = \dfrac{1 - (\tan\alpha)^2}{2 \tan\alpha},$

$$= \frac{\dfrac{1}{(\tan\alpha)^2} - 1}{\dfrac{2}{\tan\alpha}}$$

$$\therefore\ \cot 2\,\alpha = \frac{(\cot\alpha)^2 - 1}{2 \cot\alpha}.$$

88. COR. 2. $\dfrac{1}{\tan 2\,\alpha} = \dfrac{1 - (\tan\alpha)^2}{2 \tan\alpha}$

$$= \tfrac{1}{2}\left(\frac{1}{\tan\alpha} - \cancel{\frac{1}{\tan\alpha}}\right),$$ — tan α

$$\therefore\ \cot 2\,\alpha = \tfrac{1}{2}(\cot\alpha - \tan\alpha).$$

89. PROP. $Tan\ 3\,\alpha = \dfrac{3\ tan\ \alpha - (tan\ \alpha)^3}{1 - 3\,(tan\ \alpha)^2}$

For, $\tan(2\,\alpha + \alpha) = \dfrac{\tan 2\,\alpha + \tan\alpha}{1 - \tan 2\,\alpha.\ \tan\alpha}$

$$= \frac{\dfrac{2\tan\alpha}{1 - (\tan\alpha)^2} + \tan\alpha}{1 - \dfrac{2(\tan\alpha)^2}{1 - (\tan\alpha)^2}}$$

$$\therefore\ \tan 3\,\alpha = \frac{3\tan\alpha - (\tan\alpha)^3}{1 - 3(\tan\alpha)^2}.$$

90. **Prop.**

$$\left.\begin{aligned} Sin\ n\,a &= 2\,sin\,(n-1)\,a.\ cos\,a - sin\,(n-2)\,a \\ cos\ n\,a &= 2\,cos\,(n-1)\,a.\ cos\,a - cos\,(n-2)\,a \end{aligned}\right\}.$$

$$\begin{aligned} \text{For,}\quad \sin n\,a &= \sin\,\{(n-1)\,a + a\} \\ &= \sin(n-1)\,a\cos a + \cos(n-1)\,a\sin a \\ \sin(n-2)\,a &= \sin\,\{(n-1)\,a - a\} \\ &= \sin(n-1)\,a.\cos a - \cos(n-1)\,a\sin a, \\ \therefore\ \sin n\,a + \sin(n-2)\,a &= 2\sin(n-1)\,a.\ \cos a \\ \therefore\ \sin n\,a &= 2\sin(n-1)\,a.\cos a - \sin(n-2)\,a. \\ \text{Again,}\quad \cos n\,a &= \cos\,\{(n-1)\,a + a\} \\ &= \cos(n-1)\,a.\ \cos a - \sin(n-1)\,a.\ \sin a \\ \cos(n-2)\,a &= \cos\,\{(n-1)\,a - a\} \\ &= \cos(n-1)\,a.\cos a + \sin(n-1)\,a.\sin a, \\ \therefore\ \cos n\,a + \cos(n-2)\,a &= 2\cos(n-1)\,a.\ \cos a \\ \therefore\ \cos n\,a &= 2\cos(n-1)\,a.\ \cos a - \cos(n-2)\,a. \end{aligned}$$

91. Cor. 1. Let $n = 2$. And, as in (70) and (71),

$$\begin{aligned} \sin 2\,a &= 2\sin a.\ \cos a \\ \cos 2\,a &= 2\,(\cos a)^2 - 1 = 2\ \{1 - (\sin a)^2\} - 1 \\ &= 1 - 2\,(\sin a)^2. \end{aligned}$$

92. Cor. 2. Let $n = 3$. Then,

$$\begin{aligned} \sin 3\,a &= 2\sin 2\,a.\ \cos a - \sin a \\ &= 4\sin a.\ (\cos a)^2 - \sin a \\ &= 3\sin a - 4\,(\sin a)^3, \\ \cos 3\,a &= 2\cos 2\,a.\ \cos a - \cos a \\ &= 2\ \{2\,(\cos a)^2 - 1\}\ \cos a - \cos a \\ &= 4\,(\cos a)^3 - 3\cos a. \end{aligned}$$

93. **Prop.**

$$\left.\begin{aligned} Sin\,(a+\beta) + sin\,(a-\beta) &= 2\,sin\,a.\ cos\,\beta \\ sin\,(a-\beta) - sin\,(a-\beta) &= 2\,cos\,a.\ sin\,\beta \end{aligned}\right\}.$$

For, $\sin(\alpha + \beta) = \sin\alpha . \cos\beta + \cos\alpha . \sin\beta,$
$\sin(\alpha - \beta) = \sin\alpha . \cos\beta - \cos\alpha . \sin\beta,$

$\therefore$ by addition and subtraction,

$$\left.\begin{aligned}\sin(\alpha + \beta) + \sin(\alpha - \beta) &= 2\sin\alpha . \cos\beta\\ \sin(\alpha + \beta) - \sin(\alpha - \beta) &= 2\cos\alpha . \sin\beta\end{aligned}\right\}.$$

94. Cor. In the same manner,

$$\left.\begin{aligned}\cos(\alpha + \beta) + \cos(\alpha - \beta) &= 2\cos\alpha . \cos\beta\\ \cos(\alpha + \beta) - \cos(\alpha - \beta) &= -2\sin\alpha . \sin\beta\end{aligned}\right\}.$$

95. Prop.

$$\left.\begin{aligned}\mathit{Sin}\ \alpha + \sin\beta &= 2\sin\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2}\\ \sin\alpha - \sin\beta &= 2\cos\frac{\alpha + \beta}{2} . \sin\frac{\alpha - \beta}{2}\end{aligned}\right\}.$$

For, $\alpha = \frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2}$

and $\beta = \frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2}$

$$\therefore \sin\alpha = \sin\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2} + \cos\frac{\alpha + \beta}{2} . \sin\frac{\alpha - \beta}{2}$$

$$\sin\beta = \sin\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2} - \cos\frac{\alpha + \beta}{2} . \sin\frac{\alpha - \beta}{2}.$$

$\therefore$ by addition and subtraction,

$$\left.\begin{aligned}\sin\alpha + \sin\beta &= 2\sin\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2}\\ \sin\alpha - \sin\beta &= 2\cos\frac{\alpha + \beta}{2} . \sin\frac{\alpha - \beta}{2}\end{aligned}\right\}.$$

96. Prop. $$\left.\begin{aligned}\mathit{Cos}\ \alpha + \cos\beta &= 2\cos\frac{\alpha + \beta}{2} . \cos\frac{\alpha - \beta}{2}\\ \cos\alpha - \cos\beta &= -2\sin\frac{\alpha + \beta}{2} . \sin\frac{\alpha - \beta}{2}\end{aligned}\right\}.$$

For as before,

$$\cos\alpha = \cos\frac{\alpha+\beta}{2}.\ \cos\frac{\alpha-\beta}{2} - \sin\frac{\alpha+\beta}{2}.\ \sin\frac{\alpha-\beta}{2},$$

$$\cos\beta = \cos\frac{\alpha+\beta}{2}.\ \cos\frac{\alpha-\beta}{2} + \sin\frac{\alpha+\beta}{2}.\ \sin\frac{\alpha-\beta}{2}.$$

$\therefore$ by addition and subtraction,

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}.\ \cos\frac{\alpha-\beta}{2},$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}.\ \sin\frac{\alpha-\beta}{2}.$$

97. Prop. *If* $\Delta\alpha$ *represent an increment of* α*, and* $\Delta \sin\alpha$ *the corresponding increment of* $\sin\alpha$

$$\Delta \sin\alpha = 2\cos\left(\alpha + \frac{\Delta\alpha}{2}\right).\ \sin\frac{\Delta\alpha}{2}.$$

$$\sin(\alpha + \Delta\alpha) - \sin\alpha = 2\cos\frac{2\alpha + \Delta\alpha}{2}.\ \sin\frac{\Delta\alpha}{2},\ \text{(art. 95.)}$$

$$\therefore\ \Delta\sin\alpha = 2\cos\left(\alpha + \frac{\Delta\alpha}{2}\right).\ \sin\frac{\Delta\alpha}{2}.$$

98. Cor. If $\Delta\alpha$ be very small, $\sin\frac{\Delta\alpha}{2} = \frac{\Delta\alpha}{2}$

and $\cos\left(\alpha + \frac{\Delta\alpha}{2}\right) = \cos\alpha$, nearly.

$$\therefore\ \Delta\sin\alpha = \cos\alpha.\ \Delta\alpha$$

or $\dfrac{\Delta\sin\alpha}{\Delta\alpha} = \cos\alpha$, nearly.

99. Prop. $\Delta\cos\alpha = -2\sin\left(\alpha + \frac{\Delta\alpha}{2}\right).\ \sin\frac{\Delta\alpha}{2},$

For, $\cos(\alpha + \Delta\alpha) - \cos\alpha = -2\sin\dfrac{2\alpha + \Delta\alpha}{2}.\ \sin\dfrac{\Delta\alpha}{2},$

$$\therefore\ \Delta\cos\alpha = -2\sin\left(\alpha + \frac{\Delta\alpha}{2}\right).\ \sin\frac{\Delta\alpha}{2}.$$

100. Cor. If Δa be very small, as before,

$$\Delta \cos a = -\sin a \,\Delta a, \text{ nearly,}$$

$$\text{or } \frac{\Delta \cos a}{\Delta a} = -\sin a, \text{ nearly.}$$

101. Prop. $\Delta\, tan\, a = sec\, a.\ sec\, (a + \Delta a)\, sin\, \Delta a.$

$$\text{For, } \tan (a + \Delta a) - \tan a = \frac{\sin (a + \Delta a)}{\cos (a + \Delta a)} - \frac{\sin a}{\cos a}$$

$$= \frac{\sin (a + \Delta a).\ \cos a - \cos (a + \Delta a) \sin a}{\cos a.\ \cos (a + \Delta a)}$$

$$= \frac{\sin \{(a + \Delta a) - a\}}{\cos a.\ \cos (a + \Delta a)}, \text{ (art. 64.)}$$

$$= \frac{\sin \Delta a}{\cos a.\ \cos (a + \Delta a)},$$

$$\therefore \Delta \tan a = \sec a.\ \sec (a + \Delta a).\ \sin \Delta a.$$

102. Cor. $\dfrac{\Delta \tan a}{\sin \Delta a} = \sec a.\ \sec (a + \Delta a),$

$\therefore$ when Δa is very small

$$\frac{\Delta \tan a}{\Delta a} = (\sec a)^2 \text{ nearly.}$$

103. Cor. 2. These formulæ are easily changed from circular arcs to the corresponding angles. For since

$$a = \frac{\pi}{180^\circ} \cdot A$$

$$\therefore \Delta a = \frac{\pi}{180^\circ} \cdot \Delta A$$

and

$$f a = f A$$

$$\therefore \frac{\Delta \sin A}{\Delta A} = \frac{180^\circ}{\pi} \cos A$$

$$\frac{\Delta \cos A}{\Delta A} = -\frac{180^\circ}{\pi} \sin A$$

$$\frac{\Delta \tan A}{\Delta A} = \frac{180^\circ}{\pi} (\sec A)^2$$

$$\frac{\Delta \cot A}{\Delta A} = -\frac{180^\circ}{\pi} (\operatorname{cosec} A^2),$$

nearly when A is small.

104. PROP. $\dfrac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \dfrac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}.$

For, $\sin \alpha + \sin \beta = 2 \sin \frac{\alpha+\beta}{2}. \cos \frac{\alpha-\beta}{2}$,

and, $\sin \alpha - \sin \beta = 2 \cos \frac{\alpha+\beta}{2}. \sin \frac{\alpha-\beta}{2}$.

∴ by division

$$\frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{\sin \frac{\alpha+\beta}{2}}{\cos \frac{\alpha+\beta}{2}} \div \frac{\sin \frac{\alpha-\beta}{2}}{\cos \frac{\alpha-\beta}{2}},$$

$$= \frac{\tan \frac{\alpha+\beta}{2}}{\tan \frac{\alpha-\beta}{2}}.$$

105. COR. Exactly in the same manner

$$\frac{\cos \alpha + \cos \beta}{\cos \alpha - \cos \beta} = -\cot \frac{\alpha+\beta}{2}. \cot \frac{\alpha-\beta}{2}.$$

106. Many other formulæ may be derived by similar processes. As there is not the least difficulty in the operations, a table of results has been deemed all that the student will find necessary.

Values of sin 2 α.	*Values of cos 2 α.*	*Values of tan α.*
1. $2 \sin \alpha \sqrt{1 - (\sin \alpha)^2}$.	1. $1 - 2 (\sin \alpha)^2$.	1. $\dfrac{\sin 2 \alpha}{1 + \cos 2 \alpha}$
2. $2 \cos \alpha \sqrt{1 - (\cos \alpha)^2}$.	2. $2 (\cos \alpha)^2 - 1$.	2. $\dfrac{1 - \cos 2 \alpha}{\sin 2 \alpha}$.
3. $\dfrac{2 \tan \alpha}{1 + (\tan \alpha)^2}$.	3. $\dfrac{1 - (\tan \alpha)^2}{1 + (\tan \alpha)^2}$.	3. $\dfrac{\tan 2 \alpha}{\sec 2 \alpha + 1}$.
4. $\dfrac{2 \cot \alpha}{1 + (\cot \alpha)^2}$.	4. $\dfrac{(\cot \alpha)^2 - 1}{(\cot \alpha)^2 + 1}$.	4. $\dfrac{\sec 2 \alpha - 1}{\tan 2 \alpha}$.
5. $\dfrac{2\sqrt{(\sec \alpha)^2 - 1}}{(\sec \alpha)^2}$.	5. $\dfrac{2 - (\sec \alpha)^2}{(\sec \alpha)^2}$.	5. $\sqrt{\left(\dfrac{\sec 2 \alpha - 1}{\sec 2 \alpha + 1}\right)}$.
6. $\dfrac{2\sqrt{(\operatorname{cosec} \alpha)^2 - 1}}{(\operatorname{cosec} \alpha)^2}$.	6. $\dfrac{(\operatorname{cosec} \alpha)^2 - 2}{(\operatorname{cosec} \alpha)^2}$.	$\operatorname{Sec} \alpha = \sqrt{\left(\dfrac{2 \sec 2 \alpha}{\sec 2\alpha + 1}\right)}$.
		$\operatorname{Cosec} \alpha = \sqrt{\left(\dfrac{2 \sec 2 \alpha}{\sec 2\alpha - 1}\right)}$.

107. 1. $\text{Tan}\ \alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha.\ \cos\beta}$.

2. $\tan\alpha - \tan\beta = \dfrac{\sin(\alpha-\beta)}{\cos\alpha.\ \cos\beta}$.

3. $\cot\alpha + \cot\beta = \dfrac{\sin(\alpha+\beta)}{\sin\alpha.\ \sin\beta}$.

4. $\cot\alpha - \cot\beta = -\dfrac{\sin(\alpha-\beta)}{\sin\alpha.\ \sin\beta}$.

5. $\cot\alpha + \tan\beta = \dfrac{\cos(\alpha-\beta)}{\sin\alpha.\ \cos\beta}$.

6. $\cot\alpha - \tan\beta = \dfrac{\cos(\alpha+\beta)}{\sin\alpha.\ \cos\beta}$.

7. $\dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$.

8. $\dfrac{\cot\alpha + \tan\beta}{\cot\alpha - \tan\beta} = \dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$.

9. $\dfrac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = \tan\dfrac{\alpha+\beta}{2}$.

10. $\dfrac{\sin\alpha - \sin\beta}{\cos\alpha + \cos\beta} = \tan\dfrac{\alpha-\beta}{2}$.

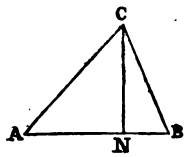
11. $\sin(\alpha+\beta).\ \sin(\alpha-\beta) = (\sin\alpha)^2 - (\sin\beta)^2$.
$= (\cos\beta)^2 - (\cos\alpha)^2$.

12. $\cos(\alpha+\beta).\ \cos(\alpha-\beta) = (\cos\alpha)^2 - (\sin\beta)^2$.
$= (\cos\beta)^2 - (\sin\alpha)^2$.

SECTION III.

ON THE SOLUTION OF PLANE TRIANGLES.

108. Prop. *In any triangle the sides are proportional to the sines of the opposite angles.*



For let ABC be the triangle. Let the angles be denoted by A, B, C, and the sides opposite to them by a, b, c, respectively. From C draw CN perpendicularly to AB. Then, since the angles at N are right angles; by (art. 30.)

$$\sin A = \frac{CN}{AC} = \frac{CN}{b}$$

$$\sin B = \frac{CN}{BC} = \frac{CN}{a}$$

$$\therefore \frac{\sin A}{\sin B} = \frac{a}{b}.$$

Similarly, $$\frac{\sin A}{\sin C} = \frac{a}{c}$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

109. Prop. *In any triangle*

$$a^2 = b^2 + c^2 - 2\,bc \cos A.$$

For in the figure of the last article

$$\frac{AN}{AC} = \cos A. \text{ (art. 31.)}$$

$$\therefore\ AN = AC.\ \cos A;$$

$$\text{But } CB^2 = CN^2 + NB^2$$

$$= CA^2 - AN^2 + (AB - AN)^2$$

$$= CA^2 + AB^2 - 2\,AB.\ AN$$

$$= CA^2 + AB^2 - 2\,AB.\ AC.\ \cos A;$$

$$\therefore\quad a^2 = b^2 + c^2 - 2\,bc \cos A.$$

110. Cor. 1. $2\,bc \cos A = b^2 + c^2 - a^2$

$$\therefore\ \cos A = \frac{b^2 + c^2 - a^2}{2\,bc}.$$

111. Cor. 2. Similarly,

$$b^2 = a^2 + c^2 - 2\,ac \cos B.$$

$$c^2 = a^2 + b^2 - 2\,ab \cos C.$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2\,ac}.$$

$$\text{and } \cos C = \frac{a^2 + b^2 - c^2}{2\,ab}.$$

112. Prop. *If* $p = b \sin A$.

$$\sin\left(45^\circ - \frac{B}{2}\right) = \sqrt{\frac{1}{2}\left(\frac{a-p}{a}\right)}.$$

$$\text{For, } \frac{\sin B}{\sin A} = \frac{b}{a}$$

$$\therefore\ \cos(90^\circ - B) = \frac{b \sin A}{a}$$

$$\therefore\ 1 - 2\left(\sin\left(45^\circ - \frac{B}{2}\right)\right)^2 = \frac{p}{a} \text{ (art. 71.)}$$

$$\therefore\ 2\left(\sin\left(45^\circ - \frac{B}{2}\right)\right)^2 = \frac{a-p}{a}$$

$$\cancel{\therefore \sin\left(45^\circ - \frac{B}{2}\right) = \sqrt{\frac{1}{2}\left(\frac{a-p}{a}\right)}}$$

113. **Prop.** *If sin* $S = \frac{2\sqrt{bc}}{b+c} \cdot \cos\frac{A}{2}$

$$a = (b+c)\cos S.$$

For
$$\begin{aligned} a^2 &= b^2 + c^2 - 2\,bc\cos A \\ &= b^2 + c^2 - 2\,bc\left\{2\left(\cos\frac{A}{2}\right)^2 - 1\right\} \\ &= b^2 + 2\,bc + c^2 - 4\,bc\left(\cos\frac{A}{2}\right)^2 \\ &= (b+c)^2 - 4\,bc\left(\cos\frac{A}{2}\right)^2 \\ &= (b+c)^2\left\{1 - \frac{4\,bc}{(b+c)^2} \cdot \left(\cos\frac{A}{2}\right)^2\right\}. \end{aligned}$$

Now the sine of an angle is never greater than unity; but $\frac{2\sqrt{bc}}{(b+c)}$, and $\cos\frac{A}{2}$ are neither of them ever greater than unity. Therefore, there is some angle whose sine is $\frac{2\sqrt{bc}}{b+c} \cdot \cos\frac{A}{2}$. Let S be this angle.

$$\begin{aligned} \therefore\ a^2 &= (b+c)^2\,\{1 - (\sin S)^2\} \\ &= (b+c)^2\,(\cos S)^2 \\ \therefore\ a\ &= (b+c) . \cos S. \end{aligned}$$

114. **Prop.** *If tan* $S = \frac{2\sqrt{bc}}{b-c} \cdot \sin\frac{A}{2}$

$$a = (b-c)\sec S.$$

For
$$\begin{aligned} a^2 &= b^2 + c^2 - 2\,bc\cos A \\ &= b^2 + c^2 - 2\,bc\left\{1 - 2\left(\sin\frac{A}{2}\right)^2\right\} \\ &= b^2 - 2\,bc + c^2 + 4\,bc\left(\sin\frac{A}{2}\right)^2 \end{aligned}$$

$$= (b-c)^2 + 4\,bc\left(\sin\frac{A}{2}\right)^2$$

$$= (b-c)^2\left\{1 + \frac{4\,bc}{(b-c)^2}\left(\sin\frac{A}{2}\right)^2\right\}.$$

Now, $\frac{2\sqrt{bc}}{b-c}$, and therefore $\frac{2\sqrt{bc}}{b-c}.\sin\frac{A}{2}$, admits of all degrees of magnitude from zero to infinity. But the tangents of angles also admit of all degrees of magnitude, therefore there is some angle whose tangent is $\frac{2\sqrt{bc}}{b-c}.\sin\frac{A}{2}$. Let S be this angle.

$$\therefore\ a^2 = (b-c)^2\ \{1 + (\tan S)^2\}$$
$$= (b-c)^2\,(\sec S)^2$$
$$\therefore\ a = (b-c)\sec S.$$

115. PROP. $c = b\ cos\ A + a\ cos\ B.$

For in the figure of art. 108.

$$AB = AN + NB$$
$$= AC.\cos A + CB.\cos B$$
$$\therefore c = b\cos A + a\cos B.$$

116. COR. $c = b\cos A + a\sqrt{\{1 - (\sin B)^2\}}$

$$= b\cos A + a\sqrt{\left\{1 - \frac{b^2}{a^2}(\sin A)^2\right\}}$$
$$= b\cos A + \sqrt{\{a^2 - b^2(\sin A)^2\}}\ .$$

117. PROP. $Tan\,\frac{A-B}{2} = \frac{a-b}{a+b}\,cot\,\frac{C}{2}.$

$$\text{For, } \frac{a}{b} = \frac{\sin A}{\sin B}$$
$$\therefore \frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{\tan \dfrac{A-B}{2}}{\tan \dfrac{A+B}{2}}$$

$$= \frac{\tan \dfrac{A-B}{2}}{\tan \left(90^\circ - \dfrac{C}{2}\right)}$$

$$\therefore \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

118. Cor. $\operatorname{Cot} \dfrac{A-B}{2} = \dfrac{a+b}{a-b} \tan \dfrac{C}{2}.$

119. Prop. *If* $\tan S = \dfrac{b}{a}$

$$\tan \frac{A-B}{2} = \tan (45^\circ - S) \cot \frac{C}{2}.$$

$$\text{For } \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\text{But } \frac{a-b}{a+b} = \frac{1-\dfrac{b}{a}}{1+\dfrac{b}{2}}$$

$$= \frac{\tan 45^\circ - \tan S}{1 + \tan 45^\circ . \tan S}$$

$$= \tan (45^\circ - S)$$

$$\therefore \tan \frac{A-B}{2} = \tan (45^\circ - S).\ \cot \frac{C}{2}.$$

120. Cor. $\operatorname{Cot} \dfrac{A-B}{2} = \dfrac{a+b}{a-b}.\ \tan \dfrac{C}{2}$

$$= \tan (45^\circ + S) \tan \frac{C}{2}.$$

121. PROP. $Sin\ \frac{A-B}{2} = \frac{a-b}{c}\ cos\ \frac{C}{2}$.

$$\text{For } \frac{a}{c} = \frac{\sin A}{\sin C}$$

$$\text{and } \frac{b}{c} = \frac{\sin B}{\sin C}$$

$$\therefore \frac{a-b}{c} = \frac{\sin A - \sin B}{\sin C}$$

$$\text{But } \sin A - \sin B = 2 \sin \frac{A-B}{2} . \cos \frac{A+B}{2} (95)$$

$$= 2 \sin \frac{A-B}{2} . \cos \left(90^\circ - \frac{C}{2}\right)$$

$$= 2 \sin \frac{A-B}{2} . \sin \frac{C}{2}$$

$$\text{and } \sin C = 2 \sin \frac{C}{2} . \cos \frac{C}{2}$$

$$\therefore \frac{a-b}{c} = \frac{\sin \frac{A-B}{2}}{\cos \frac{C}{2}}$$

$$\therefore \sin \frac{A-B}{2} = \frac{a-b}{c} . \cos \frac{C}{2}.$$

122. COR. $\frac{a+b}{c} = \frac{\sin A + \sin B}{\sin C}$

$$= \frac{\cos \frac{A-B}{2}}{\sin \frac{C}{2}}$$

$$\therefore \cos \frac{A-B}{2} = \frac{a+b}{c} \sin \frac{C}{2}.$$

123. PROP. *If* $a + b + c = 2s$

$$cos\ \frac{A}{2} = \sqrt{\left\{\frac{s.(s-a)}{b\,c}\right\}}.$$

For $a^2 = b^2 + c^2 - 2\,bc \cos A$

$$= b^2 + c^2 - 2\,bc \left\{2\left(\cos \frac{A}{2}\right)^2 - 1\right\}$$

$$\therefore\ 4\,bc\left(\cos \frac{A}{2}\right)^2 = b^2 + 2\,bc + c^2 - a^2$$
$$= (b + c)^2 - a^2$$
$$= (b + c + a)(b + c - a)$$
$$= 2\,s.\ 2\,(s - a)$$

$$\therefore \left(\cos \frac{A}{2}\right)^2 = \frac{s.\,(s - a)}{bc}$$

$$\therefore\ \cos \frac{A}{2} = \sqrt{\left\{\frac{s.\,(s - a)}{bc}\right\}}.$$

124. PROP. $Sin \frac{A}{2} = \sqrt{\left\{\frac{(s - b)\,(s - c)}{bc}\right\}}.$

For $a^2 = b^2 + c^2 - 2\,bc \cos A$

$$= b^2 + c^2 - 2\,bc\left\{1 - 2\left(\sin \frac{A}{2}\right)^2\right\}$$

$$\therefore\ 4bc\left(\sin \frac{A}{2}\right)^2 = a^2 - b^2 + 2\,bc - c^2$$
$$= a^2 - (b - c)^2$$
$$= (a - b + c)(a + b - c)$$
$$= 2\,(s - b).\ 2\,(s - c)$$

$$\therefore\ \sin \frac{A}{2} = \sqrt{\left\{\frac{(s - b).\,(s - c)}{bc}\right\}}.$$

125. COR. $\text{Tan} \frac{A}{2} = \dfrac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \sqrt{\left\{\frac{(s - b).\,(s - c)}{s.\ s - a}\right\}}.$

126. PROP. $Sin\ A = \frac{2}{bc} \sqrt{\{s.\,(s - a).\,(s - b).\,(s - c)\}}.$

For by (art. 123) $1 + \cos A = 2. \dfrac{s. (s - a)}{bc}$

and by (art. 124) $1 - \cos A = 2. \dfrac{(s - b). (s - c)}{bc}$

$$\therefore \; 1 - (\cos A)^2 = \frac{4}{b^2 c^2}. \, s. (s - a). (s - b). (s - c)$$

$$\therefore \; \sin A = \frac{2}{bc} \sqrt{\{s. (s - a). (s - b). (s - c)\}}.$$

127. Prob. *To explain an apparent imperfection in the trigonometric tables, when the angle under consideration is very small, or nearly* 90°, *or* 180°; *and to point out a method of obviating this difficulty.*

In the common tables the logarithms of the trigonometric functions are put down for *degrees* and *minutes* to 7 places of decimals. If in addition to degrees and minutes there be *seconds* in the angle under consideration, a calculation is made for them, on the supposition that the increment of the logarithmic function is proportional to the given increment of the angle.

Now, *firstly*, 7 decimals are not always sufficient.

For $\dfrac{\Delta \sin A}{\Delta A} = \dfrac{180°}{\pi} \cos A$ nearly, when ΔA is small

and $\sin A = 10^{\log \sin A}$.

Consequently, when A is about 90°, since cos A is very small, the variation for 1″ of sin A, and therefore of log sin A, is very small. And since 7 decimals just suffice for common cases, more than 7 will be necessary to distinguish between angles nearly 90°, differing by one second. That is, if log sin A be known, several angles, according to the tables, will answer to it, and it will therefore be dubious which of them is the true one.

$$\text{Also, since } \frac{\Delta \cos A}{\Delta A} = -\frac{180^\circ}{\pi} \sin A \text{ nearly,}$$

when A is very small, or nearly 180°, the determination of A from cos A, or log cos A, is involved in the same difficulty.

Secondly, the rule for calculating for the additional seconds is not always applicable.

$$\text{For } \frac{\Delta \tan A}{\sin \Delta A} = \sec A . \sec (A + \Delta A)$$

If ΔA be between 1′ and 1″, although it be true that $\sin \Delta A \propto \Delta A$ nearly: yet when A is about 90° it is not true that $\sec (A + \Delta A) = \sec A$, nearly, the variation of the secant being in this case exceedingly great. Therefore, $\frac{\Delta \tan A}{\Delta A}$ is not nearly constant, and the rule is therefore inapplicable. The same remark will apply to log sec A. Also when A is small, the rule is inapplicable for log cotan A, and log cosec A.

The calculations are supposed to have been made by the common tables. In some tables the logarithmic functions are put down for degrees, minutes, and *seconds,* and the inconvenience would be felt when *thirds* came into calculation.

The obvious method of avoiding these difficulties would be, for small angles to insert in the tables the logarithmic functions to more minute divisions than the other angles, and to a greater number of decimal places. This, however, is unnecessary. The difficulty can always be overcome by replacing the inconvenient function of A by convenient trigonometric functions of A, $\frac{A}{2}$, or $45^\circ - \frac{A}{2}$. The method of doing this will be explained in the particular cases as they occur.

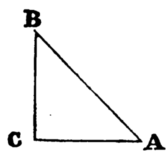
128. There are six parts in a triangle; three sides and three angles. But the three angles are always connected by

the equation $A + B + C = 180°$; so that if two be given, the third is also given. In solving the triangle, therefore, it will be considered that there are only *five* elements; and the problem is, having given *three* of them to find the remaining *two*.

129. If one of the angles C, be a right angle $A + B = 90°$. so that if one of the acute angles be given the other will also be given. There will therefore be *four* elements, and the problem will be, having given *two* of them to find the other *two*.

METHOD OF SOLVING RIGHT ANGLED TRIANGLES.

130. PROB. C *being the right angle, given a and b to find* A.



First Method.

$$\text{Tan } A = \frac{a}{b}$$

$$\therefore \log \tan A = \log a - \log b + 10.$$

Second Method.

131. $$\text{Cot } A = \frac{b}{a}$$

$$\therefore \log \cot A = \log b - \log a + 10.$$

The first method applies A is very small, or b much greater than a; the second when A is nearly 90°, or a much greater than b. (See article 127.)

132. **Prob.** *Given a and b to find c.*

First Method.

$$c = \sqrt{(a^2 + b^2)}$$
$$\therefore \log c = \tfrac{1}{2} \log (a^2 + b^2).$$

Second Method.

133. If a and b consist of many figures, the operation of squaring will be very laborious. This might be done by a table of logarithms. It will, however, be more easy in this case to find A by the last article, and then determine c thus.

$$\frac{a}{c} = \sin A$$
$$\therefore c = \frac{a}{\sin A}$$
$$\therefore \log c = \log a - \log \sin A + 10.$$

134. **Prob.** *Given b and c to find a.*

$$a = \sqrt{(c^2 - b^2)}$$
$$= \sqrt{\{(c + b)(c - b)\}}$$
$$\therefore \log a = \tfrac{1}{2} \log (c + b) + \tfrac{1}{2} \log (c - b).$$

135. PROB. *Given b and c to find* A.

First Method.

$$\cos A = \frac{b}{c}$$

$$\therefore \log \cos A = \log b - \log c + 10.$$

Second Method.

136. $$(\text{Sin } A)^2 = 1 - (\cos A)^2$$
$$= 1 - \frac{b^2}{c^2}$$
$$= \frac{(c+b)(c-b)}{c^2}$$
$$\therefore \sin A = \frac{\sqrt{\{(c+b)(c-b)\}}}{c}$$
$$\therefore \log \sin A = \tfrac{1}{2}\{\log(c+b) + \log(c-b)\} - \log c + 10.$$

The first method applies when A is nearly 90°, or b is much less than c; the second when A is small, or b nearly equal to c.

137. PROB. *Given c and* A *to find a.*

$$\frac{a}{c} = \sin A$$

$$\therefore a = c \sin A$$

$$\therefore \log a = \log c + \log \sin A - 10.$$

138. PROB. *Given b and* A *to find a.*

First Method.

$$\frac{a}{b} = \tan A$$

$$\therefore a = b \tan A$$

$$\therefore \log a = \log b + \log \tan A - 10.$$

Second Method.

139. $$\frac{b}{a} = \cot A$$

$$\therefore a = \frac{b}{\cot A}$$

$$\therefore \log a = \log b - \log \cot A + 10.$$

The first method is to be used when A is less, and the second when A is greater than 45°. See articles 102. and 127.

140. Prob. *Given b and A to find c.*

$$\frac{b}{c} = \cos A$$

$$\therefore c = \frac{b}{\cos A}$$

$$\therefore \log c = \log b - \log \cos A + 10.$$

SOLUTION OF OBLIQUE ANGLED TRIANGLES.

141. Prob. *Given two sides b and c, and the included angle A to find the opposite side a.*

First Method.

$$\sin S = \frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}$$

$$a = (b + c)\cos S,$$

or, in logarithms,

$$\log \sin S = \log 2 + \tfrac{1}{2}\log b + \tfrac{1}{2}\log c + \log\cos\frac{A}{2} - \log(b+c)$$

$$\log a = \log(b + c) + \log\cos S - 10.$$

Second Method.

142. $$\text{Tan S} = \frac{2\sqrt{bc}}{b-c} \sin\frac{A}{2}$$

$$a = (b-c). \quad \sec S,$$

or $$\log\tan S = \log 2 + \tfrac{1}{2}\log b + \tfrac{1}{2}\log c + \log\sin\frac{A}{2} \log(b-c)$$

$$\log a = \log(b-c) + \log\sec S - 10.$$

The first method must not be used when one of the sides is nearly equal to the other and the angle of inconsiderable magnitude. For $2\sqrt{bc}$ is less than $b + c$ by $(\sqrt{b} - \sqrt{c})^2$: this will be very small when b is nearly equal to c, and therefore $\frac{2\sqrt{bc}}{b+c}. \cos\frac{A}{2}$ will be nearly equal to unity, and the subsidiary angle S will not be determined accurately.

The second method must not be used when one of the sides is nearly equal to the other, and the angle A of considerable magnitude. For in that case $b - c$ will be very small, and therefore $\frac{2\sqrt{bc}}{b-c} \sin\frac{A}{2}$ will be very much greater than unity, and therefore the angle S will not be determined accurately.

143. PROB. *Given two sides a and b, and an angle* A *opposite to one of them to find the remaining angles* B *and* C.

First Method.

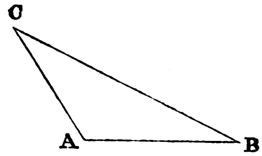
$$\text{Sin B} = \frac{b}{a} \sin A$$

$$\therefore \log\sin B = \log b - \log a + \log\sin A$$

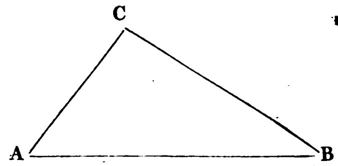
$$\text{and } C = 180^\circ - (A + B).$$

But since $\sin B = \sin(180^\circ - B)$ it *may* be dubious whether the angle in the tables, or its supplement, be the pro-

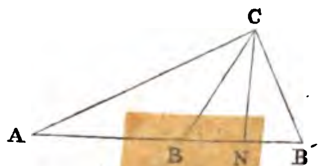
per angle. This is called the ambiguous case; it is, however, in reality, ambiguous only when A is less than 90°, and a less than b. For



1. When A is greater than 90°, sine A + B + C = 180°, B must be less than 90°, and the angle given in the tables is the proper one, as in figure 1.



2. When A is less than 90°, and a greater than b, therefore the angle A is greater than B, or B is less than 90°, as in the first case. See figure 2.



3. But when A is less in 90°, and a less than b, therefore B is greater than A. But either B, or 180° — B, will satisfy this condition; as in figure 3, where CB = CB′. In this case, it is doubtful whether CBA, or CB′A = CBB′ = 180° — CBA be the proper angle.

Second Method.

144. If the angle B be nearly 90° it cannot be found accurately by the common tables, but by (art. 112.)

$$\text{If } p = b \sin A$$

$$\sin\left(45° - \frac{B}{2}\right) = \sqrt{\tfrac{1}{2}\left(\frac{a-p}{a}\right)}$$

$$\therefore \log p = \log b + \log \sin A - 10$$

$$\log \sin\left(45° - \frac{B}{2}\right) = \tfrac{1}{2}\,\{\log (a-p) - \log a - \log 2\} + 10.$$

And since B is nearly 90°, $45° - \dfrac{B}{2}$ will be very small, and will be determined accurately by the sine.

145. Prob. *Gain two sides a and b, and an angle* A *opposite to one of them to find the remaining side c.*

First Method.

$$c = b \cos A \pm \sqrt{\{a^2 - b^2 (\sin A)^2\}}.$$

This formula cannot be easily adapted to logarithmic computation, but it affords an algebraical method of explaining the ambiguous case which is here denoted by the sign $\pm$.

Since an angle of a triangle is never greater than 180°, therefore the sines of the angles are always positive. And the sides are proportional to the sines of the opposite angles, therefore the sides have all the same algebraic sign. Hence a and b being positive, c also must be positive.

1. If A be greater than 90°, $b \cos A$ is negative, and c is positive, therefore the positive sign must be used.

2. If A be less than 90°, and b be less than a

$\therefore \sqrt{\{a^2 - b^2 (\sin A)^2\}}$ which $= \sqrt{\{a^2 - b^2 + b^2 (\cos A)^2\}}$

is greater than b cos A, therefore the positive sign must be used, for otherwise c would be negative.

3. But if A be less than 90°, and b greater than a

$$\sqrt{\{a^2 - b^2 (\sin A)^2\}} \text{ is less than } b \cos A,$$

and therefore either sign may be taken, as in fig. 3, of (art. 143.) where the third side is $AB = AN - NB$

$$\text{or } AB' = AN + NB.$$

Second Method.

146. The angle C must be determined by articles 143, or 144, and then since

$$\frac{c}{a} = \frac{\sin C}{\sin A}$$

$$\therefore c = a . \frac{\sin C}{\sin A}$$

$$\therefore \log c = \log a + \log \sin C - \log \sin A.$$

147. PROB. *Given the three sides to determine the three angles.*

First Method.

$$\text{Sin } A = \frac{2}{bc} \sqrt{\{s . (s - a) . (s - b) . (s - c)\}}$$

$$\therefore \log \sin A = \tfrac{1}{2} \{\log s + \log (s - a) + \log (s - b) + \log (s - c)\} + \log 2 - \log b - \log c + 10.$$

This method is inconvenient when A is nearly 90°.

Second Method.

148. $\quad \text{Sin } \frac{A}{2} = \sqrt{\left\{\frac{(s - b)(s - c)}{bc}\right\}}$

$$\therefore \log \sin \frac{A}{2} = \tfrac{1}{2}\,\{\log (s-b) + \log (s-c) - \log b - \log c\} + 10.$$

This method is inconvenient when A is nearly 180°.

Third Method.

149. $$\operatorname{Cos} \frac{A}{2} = \sqrt{\left\{\frac{s.\ (s-c)}{bc}\right\}}$$

$$\therefore \log \cos \frac{A}{2} = \tfrac{1}{2}\,\{\log s + \log (s-c) - \log b - \log c\} + 10.$$

This method is inconvenient when A is small. Either the second or third method may be adopted when A is nearly 90°. When A is less than 90°, $\frac{A}{2}$ will be a little less than 45°, therefore the variation of $\sin \frac{A}{2}$ will be greater than that of $\cos \frac{A}{2}$, and some advantage will be derived by using the second method. On the contrary, when A is greater than 90°, the third method is preferable.

150. PROB. *Given two sides a and b and the included angle* C *to find the remaining angles* A *and* B.

First Method.

A − B must be found from the formula

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}, \text{ or}$$

$$\log \tan \frac{A-B}{2} = \log (a-b) - \log (a+b) + \log \cot \frac{C}{2}$$

and A + B = 180° − C is known ∴ A and B are known.

Second Method.

151. $\operatorname{Cot}\frac{A-B}{2} = \frac{a+b}{a-b}\tan\frac{C}{2}$, or

$$\log\cot\frac{A-B}{2} = \log(a+b) - \log(a-b) + \log\tan\frac{C}{2}.$$

152. PROB. *Given the three sides to find the area.*

$$\begin{aligned}\text{Area} &= \tfrac{1}{2}\,(\text{base})\,(\text{perpendicular})\\ &= \tfrac{1}{2}\,c.\ b\sin A\\ &= \sqrt{\{s.\ (s-a).\ (s-b).\ (s-c)\}}.\ (126.)\end{aligned}$$

SECTION IV.

ON MULTIPLE ARCS.

153. Prop. *If* $2 \cos a = x + \frac{1}{x}$

$$2\sqrt{-1}\,\sin a = x - \frac{1}{x}.$$

For
$$-4(\sin a)^2 = -4 + 4(\cos a)^2$$
$$= -4 + \left(x + \frac{1}{x}\right)^2$$
$$= x^2 - 2 + \frac{1}{x^2}$$
$$= \left(x - \frac{1}{x}\right)^2$$

$$\therefore\ 2\sqrt{-1}\,\sin a = x - \frac{1}{x}.$$

154. Cor. 1. In the same manner if $2 \cos a = x^{\frac{m}{n}} + \frac{1}{x^{\frac{m}{n}}}$;

then also $2\sqrt{-1}\,\sin a = x^{\frac{m}{n}} - \frac{1}{x^{\frac{m}{n}}}$.

155. Cor. 2. By addition and subtraction

$$\cos a + \sqrt{-1}\,\sin a = x$$
$$\cos a - \sqrt{-1}\,\sin a = \frac{1}{x}.$$

156. PROP. *If* $2 \cos \alpha = x + \frac{1}{x}$, *then*

$$2 \cos \frac{m}{n} \alpha = x^{\frac{m}{n}} + \frac{1}{x^{\frac{m}{n}}}$$

$$\text{and } 2\sqrt{-1} \sin \frac{m}{n} \alpha = x^{\frac{m}{n}} - \frac{1}{x^{\frac{m}{n}}}.$$

Firstly, When the index is a positive integer.
Suppose it true, that

$$2 \cos m \alpha = x^m + \frac{1}{x^m}$$

$$\therefore 2\sqrt{-1} \sin m \alpha = x^m - \frac{1}{x^m} \quad (154.)$$

$$\therefore 4 \cos (m+1) \alpha = 2 \cos m \alpha .\ 2 \cos \alpha - 2 \sin m \alpha .\ 2 \sin \alpha$$

$$= 2 \cos m \alpha .\ 2 \cos \alpha + 2\sqrt{-1} \sin m \alpha .\ 2\sqrt{-1} \sin \alpha$$

$$= \left(x^m + \frac{1}{x^m}\right)\left(x + \frac{1}{x}\right) + \left(x^m - \frac{1}{x^m}\right)\left(x - \frac{1}{x}\right)$$

$$= 2 x^{m+1} + \frac{2}{x^{m+1}}$$

$$\therefore 2 \cos (m+1) \alpha = x^{m+1} + \frac{1}{x^{m+1}}$$

$$\text{and } \therefore 2\sqrt{-1} \sin . \alpha\ (m+1) \alpha = x^{m+1} - \frac{1}{x^{m+1}} \quad (154.)$$

If therefore the theorem be true for $m \alpha$, it is true for $(m+1) \alpha$. But it is supposed, that x is of such a value that it is true for α, therefore it is true for 2α, and consequently for 3α; and so on, by induction, it is true for any positive integer.

Secondly, When the index is a negative integer

$$2 \cos (-m \alpha) = 2 \cos m \alpha$$

$$= x^m + \frac{1}{x^m}$$

$$= x^{-m} + \frac{1}{x^{-m}}.$$

Thirdly, When the index is a positive, or negative fraction.

Let $\frac{m}{n}$ be the index, m and n being integers.

Let $2 \cos \frac{m}{n} \alpha = y + \frac{1}{y}$

$$\left.\begin{aligned} \therefore\ 2 \cos m\, \alpha &= y^n + \frac{1}{y^n} \\ \text{But } 2 \cos m\, \alpha &= x^m + \frac{1}{x^m} \end{aligned}\right\} \text{by the first and second cases.}$$

$$\therefore\ y^n = x^m$$

$$\therefore\ y = x^{\frac{m}{n}}$$

$$\therefore\ 2 \cos \frac{m}{n}\, \alpha = x^{\frac{m}{n}} + \frac{1}{x^{\frac{m}{n}}}$$

and $\therefore\ 2 \sqrt{-1} \sin \frac{m}{n}\, \alpha = x^{\frac{m}{n}} - \frac{1}{x^{\frac{m}{n}}}$.

157. COR. 1. If $2 \cos \alpha = x + \frac{1}{x}$ and $2 \cos \beta = y + \frac{1}{y}$

$$4 \cos(\alpha+\beta) = 2 \cos \alpha .\ 2 \cos \beta + 2\sqrt{-1} \sin \alpha .\ 2\sqrt{-1} \sin \beta$$

$$= \left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + \left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)$$

$$= 2\, xy + \frac{2}{xy}$$

$$\therefore\ 2 \cos(\alpha + \beta) = xy + \frac{1}{xy}$$

and $\therefore\ 2 \sqrt{-1} \sin(\alpha + \beta) = xy - \frac{1}{xy}$.

158. COR. 2. $2 \cos(\alpha - \beta) = \frac{x}{y} + \frac{y}{x}$

and $2 \sqrt{-1} \sin(\alpha - \beta) = \frac{x}{y} - \frac{y}{x}$.

159. PROP.

$$(Cos\ \alpha \pm \sqrt{-1}\ sin\ \alpha)^{\frac{m}{n}} = cos\ \frac{m}{n}\alpha \pm \sqrt{-1}\ sin\ \frac{m}{n}\ \alpha.$$

Firstly, When the index is a positive integer.

Suppose it true, that

$$(\cos x \pm \sqrt{-1} \sin \alpha)^m = \cos m\,\alpha \pm \sqrt{-1}\, \sin m\,\alpha$$

$\therefore$ by multiplying by $\cos \alpha \pm \sqrt{-1} \sin \alpha$

$$\begin{aligned}(\cos \alpha \pm \sqrt{-1} \sin \alpha)^{m+1} &= \cos m\,\alpha .\ \cos \alpha - \sin m\,\alpha .\ \sin \alpha \\ &\quad \pm \sqrt{-1}\,(\sin m\,\alpha .\ \cos \alpha + \cos m\,\alpha .\ \sin \alpha) \\ &= \cos (m+1)\,\alpha \pm \sqrt{-1} \sin (m+1)\,\alpha .\end{aligned}$$

If therefore the theorem be true for $m\,\alpha$, it is true for $(m + 1)\,\alpha$. But it is true when $m = 1$, for then both sides of the equation coincide, therefore it is true for $m = 2$, and therefore for $m = 3$; and so on, by induction, it is true for any positive integer.

Secondly, When the index is a negative integer.

$$\begin{aligned}(\cos \alpha \pm \sqrt{-1} \sin \alpha).\quad (\cos \alpha \mp \sqrt{-1} \sin \alpha &\\ = (\cos \alpha)^2 + (\sin \alpha)^4 &= 1\end{aligned}$$

$$\therefore (\cos \alpha \pm \sqrt{-1} \sin \alpha)^{-1} = \cos \alpha \mp \sqrt{-1} \sin \alpha$$

$$\begin{aligned}\therefore (\cos \alpha \pm \sqrt{-1} \sin \alpha)^{-m} &= (\cos \alpha \mp \sqrt{-1} \sin \alpha)^m \\ &= \cos m\,\alpha \mp \sqrt{-1} \sin m\,\alpha \\ &= \cos(-m\alpha) \pm \sqrt{-1} \sin(-m\alpha).\end{aligned}$$

Thirdly, When the index is a positive, or negative fraction.

Let the index be $\frac{m}{n}$, m and n being integers.

$$\begin{aligned}\left(\cos \frac{m}{n}\alpha \pm \sqrt{-1} \sin \frac{m}{n}\alpha\right)^n &= \cos m\,\alpha \pm \sqrt{-1} \sin m\,\alpha \\ &= (\cos \alpha \pm \sqrt{-1} \sin \alpha)^m\end{aligned}$$

by the first and second cases,

$$\therefore (\cos \alpha \pm \sqrt{-1} \sin \alpha)^{\frac{m}{n}} = \cos \frac{m}{n}\alpha \pm \sqrt{-1} \sin \frac{m}{n}\alpha.$$

Another Proof.

By (art. 156.)

$$2 \cos \frac{m}{n}\alpha = x^{\frac{m}{n}} + \frac{1}{x^{\frac{m}{n}}}$$

$$2\sqrt{-1}\sin\frac{m}{n}\alpha = x^{\frac{m}{n}} - \frac{1}{x^{\frac{m}{n}}}$$

$\therefore$ by adding and dividing by 2

$$\cos\frac{m}{n}\alpha + \sqrt{-1}\sin\frac{m}{n}\alpha = x^{\frac{m}{n}}$$

$$= (\cos\alpha + \sqrt{-1}\sin\alpha)^{\frac{m}{n}} \text{ by (art. 155.)}$$

160. Cor. 1. $(\cos\alpha + \sqrt{-1}\sin\alpha).(\cos\beta + \sqrt{-1}\sin\beta)$
$= \cos(\alpha+\beta) + \sqrt{-1}\sin(\alpha+\beta)$.

161. Cor. 2. Since $(\cos\beta - \sqrt{-1}\sin\beta) = \dfrac{1}{\cos\beta + \sqrt{-1}\sin\beta}$

$$\therefore \frac{\cos\alpha + \sqrt{-1}\sin\alpha}{\cos\beta + \sqrt{-1}\sin\beta} = \cos(\alpha-\beta) + \sqrt{-1}\sin(\alpha-\beta).$$

162. Prop.

If $T = 1 - \dfrac{m.(m-1)}{1.\ 2}(tan\,\alpha)^2 + \dfrac{m.(m-1).(m-2).(m-3)}{1.\ 2.\ 3.\ 4}(tan\,\alpha)^4 ..$

and $T' = m\,tan\,\alpha - \dfrac{m.(m-1).(m-2)}{1.\ 2.\ 3}(tan\,\alpha)^3 + \ldots$

Then, when m is a positive integer,

$$cos\ m\,\alpha = (cos\,\alpha)^m\,T$$
$$and\ sin\ m\,\alpha = (cos\,\alpha)^m\,T'.$$

For $\cos m\alpha + \sqrt{-1}\sin m\alpha = (\cos\alpha + \sqrt{-1}\sin\alpha)^m$

$$= (\cos\alpha)^m\left(1+\sqrt{-1}\frac{\sin\alpha}{\cos\alpha}\right)^m$$
$$= (\cos\alpha)^m(1+\sqrt{-1}\tan\alpha)^m$$
$$= (\cos\alpha)^m(T+\sqrt{-1}\,T')$$

by expanding by the binomial theorem.

But because m is a positive integer $(\cos\alpha)^m$ cannot involve any imaginary quantity. Therefore by equating real and imaginary quantities

$$\cos m\alpha = (\cos\alpha)^m\,T$$
$$\sin m\alpha = (\sin\alpha)^m\,T'.$$

163. Cor. 1. By division

$$\tan m\,a = \frac{T'}{T}\cdot$$

164. Cor. 2. By multiplication

$$\cos m\,a = (\cos a)^m - \frac{m.\,(m-a)}{1.\ 2}\,(\cos a)^{m-2}\,(\sin a)^2 + \ldots$$

$$\sin m\,a = m\,(\cos a)^{m-1}\,\sin a - \frac{m.\,(m-1).\,(m-2)}{1.\ 2.\ 3}(\cos a)^{m-3}(\sin a)^2 + \ldots$$

165. Prop. $Cos\ a = 1 - \frac{a^2}{1.\ 2} + \frac{a^4}{1.\ 2.\ 3.\ 4}\ldots$

$$sin\ a = a - \frac{a^3}{1.\ 2.\ 3} + \frac{a^5}{1.\ 2.\ 3.\ 4.\ 5} - \ldots\ldots$$

For $\cos a = \cos m\,\frac{a}{m} = \left(\cos\frac{a}{m}\right)^m \left\{1 - \frac{m.\,(m-1)}{1.\ 2}\left(\tan\frac{a}{m}\right)^2 + \right.$

$$\left.\frac{m.\,(m-1).\,(m-2).\,(m-3)}{1.\ 2.\ 3.\ 4}\left(\tan\frac{a}{m}\right)^4 - \ldots\right\}$$

$$= \left(\cos\frac{a}{m}\right)^m \left\{1 - \left(\frac{1}{2} - \frac{1}{2m}\right)\left(m\tan\frac{a}{m}\right)^2 + \right.$$

$$\left.\left(\frac{1}{2} - \frac{1}{2m}\right)\left(\frac{1}{3} - \frac{2}{3m}\right).\left(\frac{1}{4} - \frac{3}{4m}\right)\left(m\tan\frac{a}{m}\right)^4 - \ldots\right\}.$$

Now as m is increased $\frac{a}{m}$ will be diminished, and $\tan\frac{a}{m}$ will continually approximate to $\frac{a}{m}$, and $m\tan\frac{a}{m}$ to a. (1.)

Let $m = \infty$, then $\left(\cos\frac{a}{m}\right)^m = 1$, $m\tan\frac{a}{m} = a$, and $\frac{1}{2m}$, $\frac{2}{3m}$, $\frac{3}{4m}\ldots$ will vanish.

$$\therefore \cos a = 1 - \frac{a^2}{1.\ 2} + \frac{a^4}{1.\ 2.\ 3.\ 4} - \ldots$$

Similarly,

$$\sin a = a - \frac{a^3}{1.\ 2.\ 3} + \frac{a^5}{1.\ 2.\ 3.\ 4.\ 5}\ldots$$

166. Cor. 1.

$$\text{Cos}\,a+\sqrt{-1}\sin a=1+a\sqrt{-1}+\frac{(a\sqrt{-1})^2}{1.\ 2}+\frac{(a\sqrt{-1})^3}{1.\ 2.\ 3}+\ldots$$
$$=e^{a\sqrt{-1}}.$$

167. Cor. 2. $\text{Cos}\,a+\sqrt{-1}\sin a=e^{a\sqrt{-1}}$

$$\therefore \quad \cos a-\sqrt{-1}\sin a=e^{-a\sqrt{-1}}$$
$$\therefore\ 2\cos a=e^{a\sqrt{-1}}+e^{-a\sqrt{-1}}$$
$$\text{and } 2\sqrt{-1}\sin a=e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}.$$

168. All these formulæ may be easily obtained by aid of the differential calculus.

$$d.\cos a=-\sin a\,d\,a$$
$$=\sqrt{-1}\sin a.\ d\,a\sqrt{-1}$$
$$\sqrt{-1}\,d.\sin a=\cos a\,d\,a\sqrt{-1}$$
$$\therefore d.\cos a+\sqrt{-1}\,d.\sin a=(\cos a+\sqrt{-1}\sin a)\,d\,a\sqrt{-1}$$
$$\therefore \frac{d.\cos a+\sqrt{-1}\,d\sin a}{\cos a+\sqrt{-1}\sin a}=d.\ a\sqrt{-1}$$
$$\therefore \log_e(\cos a+\sqrt{-1}\sin a)=a\sqrt{-1}+\text{C}$$

by making $a=0$, it appears that $\text{C}=0$

$$\therefore \log_e(\cos a+\sqrt{-1}\sin a)=a\sqrt{-1}$$
$$\therefore \cos a+\sqrt{-1}\sin a=e^{a\sqrt{-1}}$$
$$\therefore \cos\frac{m}{n}a+\sqrt{-1}\sin\frac{m}{n}a=e^{\frac{m}{n}a\sqrt{-1}}$$
$$=(e^{a\sqrt{-1}})^{\frac{m}{n}}$$
$$=(\cos a+\sqrt{-1}\sin a)^{\frac{m}{n}},$$

$$\text{Also } \cos a-\sqrt{-1}\sin a=e^{-a\sqrt{-1}}$$
$$\therefore\ 2\cos a=e^{a\sqrt{-1}}+e^{-a\sqrt{-1}}$$
$$\text{and } 2\sqrt{-1}\sin a=e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}.$$

169. Prop.

$$Cos\ a=\left(1-\frac{2^2\,a^2}{\pi^2}\right)\left(1-\frac{2^2\,a^2}{3^2\,\pi^2}\right)\left(1-\frac{2^2\,a^2}{5^2\,\pi^2}\right)\cdots$$

$$\sin\, a = a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\,\pi^2}\right)\left(1 - \frac{a^2}{3^2\,\pi^2}\right)\ldots\ldots$$

By assuming a equal to any of the arcs

$$\pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \&c.$$

in every case $\cos a = 0$,

$$\therefore\ a - \frac{\pi}{2},\ a + \frac{\pi}{2},\ a - \frac{3\pi}{2},\ a + \frac{3\pi}{2},\ \ldots\ldots$$

and therefore

$$1 - \frac{2a}{\pi},\ 1 + \frac{2a}{\pi},\ 1 - \frac{2a}{3\pi},\ 1 + \frac{2a}{3\pi},\ \ldots\ldots$$

are simple factors of $\cos a$,

$$\therefore\ \cos a = c\left(1 - \frac{2^2\,a^2}{\pi^2}\right)\left(1 - \frac{2^2\,a^2}{3^2\,\pi^2}\right)\ldots\ldots$$

where c = some constant quantity.

By making $a = 0$, $c = 1$

$$\therefore\ \cos a = \left(1 - \frac{2^2\,a^2}{\pi^2}\right)\left(1 - \frac{2^2\,a^2}{3^2\,\pi^2}\right)\ldots\ldots$$

Again, by assuming a equal to any of the arcs

$$0,\ \pm\,\pi,\ \pm\,2\,\pi,\ \pm\,3\,\pi,\ \ldots\ldots$$

in every case $\sin a = 0$.

Therefore, as before,

$$\sin a = c\,a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\,\pi^2}\right) + \ldots\ldots$$

But when $a = 0$, $\frac{\sin a}{a} = 1\ \therefore\ c = 1$

$$\therefore\ \sin a = a\left(1 - \frac{a^2}{\pi^2}\right).\left(1 - \frac{a^2}{2^2\,\pi^2}\right)\ldots\ldots$$

170. COR. 1. In the expression,

$$\sin a = a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\,\pi^2}\right)\left(1 - \frac{a^2}{3^2\,\pi^2}\right)\ldots\ldots$$

Let $a = \frac{\pi}{2}$

$$\therefore 1 = \frac{\pi}{2}\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{4^2}\right)\left(1 - \frac{1}{6^2}\right)\dots$$

$$= \frac{\pi}{2}\cdot\frac{2^2 - 1^2}{2^2}\cdot\frac{4^2 - 1^2}{4^2}\cdot\frac{6^2 - 1^2}{6^2}\dots$$

$$= \frac{\pi}{2}\cdot\frac{1.3}{2^2}\cdot\frac{3.5}{4^2}\cdot\frac{5.7}{6^2}\dots$$

$$= \frac{\pi}{2}\cdot\frac{3^2.\,5^2.\,7^2\dots}{2^2.\,4^2.\,6^2\dots}$$

$$\therefore \frac{\pi}{2} = \frac{2^2.\,4^2.\,6^2\dots}{3^2.\,5^2.\,7^2\dots}$$

This expression was first given by Wallis.

171. Prop.

$$Tan\ (\alpha+\beta+\gamma+\dots) = \frac{\Sigma\,(tan\ \alpha) - \Sigma(tan\ \alpha.\ tan\ \beta.\ tan\ \gamma.) + \dots}{1 - \Sigma\ (tan\ \alpha.\ tan\ \beta) + \dots}$$

Where $\Sigma\ (tan\ \alpha) = tan\ \alpha + tan\ \beta + tan\ \gamma + \dots$

$\Sigma\ (tan\ \alpha.\ tan\ \beta) = tan\ \alpha.\ tan\ \beta + tan\ \alpha.\ tan\ \gamma + \dots$

$\Sigma\,(tan\ \alpha.\ tan\ \beta.\ tan\ \gamma) = tan\ \alpha.\ tan\ \beta.\ tan\ \gamma + \dots = \dots$

For, by continued multiplication and reduction,

$$\cos\,(\alpha+\beta+\gamma+\dots) + \sqrt{-1}\,\sin\,(\alpha+\beta+\gamma+\dots)$$

$$= (\cos\alpha + \sqrt{-1}\sin\alpha).\,(\cos\beta + \sqrt{-1}\sin\beta).\,(\cos\gamma + \sqrt{-1}\sin\gamma)\dots$$

$$= (\cos\alpha.\cos\beta.\cos\gamma\dots)\ \{(1 + \sqrt{-1}\tan\alpha).\,(1 + \sqrt{-1}\tan\beta).\,(1 + \sqrt{-1}\tan\gamma)\dots\}$$

$$= (\cos\alpha.\cos\beta.\cos\gamma\dots)\ \{1 + \sqrt{-1}\,\Sigma\,(\tan\alpha) - \Sigma(\tan\alpha.\tan\beta) - \sqrt{-1}\,\Sigma\,(\tan\alpha.\tan\beta.\tan\gamma) + \dots\}.$$

Therefore, by equating real and imaginary quantities,

$$\cos\,(\alpha+\beta+\gamma+\dots) = \cos\alpha.\cos\beta.\cos\gamma\dots\ \{1 - \Sigma\,(\tan\alpha.\tan\beta) + \dots\}$$

$$\sin\,(\alpha+\beta+\gamma+\dots) = \cos\alpha.\cos\beta.\cos\gamma\dots\ \{\Sigma\,(\tan\alpha) - \Sigma\,(\tan\alpha.\tan\beta.\tan\gamma) + \dots\}$$

$$\therefore \tan\,(\alpha+\beta+\gamma+\dots) = \frac{\Sigma(\tan\alpha) - \Sigma(\tan\alpha.\tan\beta.\tan\gamma) + \dots}{1 - \Sigma\,(\tan\alpha.\tan\beta) + \dots}.$$

172. Cor. 1. If $\alpha + \beta + \gamma = i\,\pi$

i being any integer,

$$\tan(\alpha + \beta + \gamma) = 0$$

$$\therefore \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha.\ \tan\beta.\ \tan\gamma.$$

173. Cor. 2. If $\alpha + \beta + \gamma = (i + \frac{1}{2})\,\pi$

$$\tan(\alpha + \beta + \gamma) = \infty$$

$$\therefore \tan\alpha.\ \tan\gamma + \tan\alpha.\ \tan\gamma + \tan\beta.\ \tan\gamma = 1.$$

174. Cor. 3. If $\alpha + \beta + \gamma = (i + \frac{1}{4})\,\pi$

$$\tan(\alpha + \beta + \gamma) = 1$$

$$\therefore \tan\alpha.\ \tan\beta.\ \tan\gamma - \Sigma(\tan\alpha.\ \tan\beta) - \Sigma(\tan\alpha) = 1.$$

175. Prop. *The n values of* $[\cos\alpha \pm \sqrt{-1}\sin\alpha]^{\frac{m}{n}}$ *may be obtained from the formula*

$$\cos\frac{m}{n}(2\,i\,\pi + \alpha) \pm \sqrt{-1}\sin\frac{m}{n}(2\,i\,\pi + \alpha),$$

by substituting for i the integers from 0 *to* $n - 1$.*

For, by (art. 159), it was shown that

$$(\cos\alpha \pm \sqrt{-1}\sin\alpha)^{\frac{m}{n}} = \cos\frac{m}{n}\alpha \pm \sqrt{-1}\sin\frac{m}{n}\alpha.$$

But there are n values of $[\cos\alpha \pm \sqrt{-1}\sin\alpha]^{\frac{m}{n}}$.

* In this and the following propositions, when a quantity raised to a fractional power is inclosed by the ordinary parenthesis ( ), the meaning is restricted to the *direct* or *arithmetic* value of the function. If any whatever of the values which that function is capable of assuming is intended, the brackets [ ] are used.

Also, if $2\,i\,\pi + \alpha$ be substituted for α, i being an integer, the first member of the equation remains unchanged, and the formula becomes

$$[\cos\alpha \pm \sqrt{-1}\sin\alpha]^{\frac{m}{n}} = \cos\frac{m}{n}(2i\,\pi+\alpha) \pm \sqrt{-1}\sin\frac{m}{n}(2i\pi+\alpha).$$

Therefore, by substituting for i the integers 0, 1, &c. $n-1$, the values of $[\cos\alpha \pm \sqrt{-1}\sin\alpha]^{\frac{m}{n}}$ are

$$\cos\frac{m}{n}\alpha \pm \sqrt{-1}\sin\alpha$$

$$\cos\frac{m}{n}(2\pi+\alpha) \pm \sqrt{-1}\sin\frac{m}{n}(2\pi+\alpha)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\cos\frac{m}{n}\{2(n-1)\pi+\alpha\} \pm \sqrt{-1}\sin\frac{m}{n}\{2(n-1)\pi+\alpha).$$

After this the terms will recur for

$$\cos\frac{m}{n}(2n\pi+\alpha) \pm \sqrt{-1}\sin\frac{m}{n}(2n\pi+\alpha)$$

$$= \cos\left(2m\pi+\frac{m}{n}\alpha\right) \pm \sqrt{-1}\sin\left(2m\pi+\frac{m}{n}\alpha\right)$$

$$= \cos\frac{m}{n}\alpha \pm \sqrt{-1}\sin\frac{m}{n}\alpha.$$

Thus may it be shown that the succeeding terms recur.

176. Cor. 1. If the sign of α be changed the result is

$$[\cos\alpha \mp \sqrt{-1}\sin\alpha]^{\frac{m}{n}} = \cos\frac{m}{n}(2i\pi-\alpha) \pm \sqrt{-1}\sin\frac{m}{n}(2i\pi-\alpha),$$

or,

$$[\cos\alpha \pm \sqrt{-1}\sin\alpha]^{\frac{m}{n}} = \cos\frac{m}{n}(2i\pi-\alpha) \mp \sqrt{-1}\sin\frac{m}{n}(2i\,\pi-\alpha).$$

It must not be concluded from this that

$$\cos\frac{m}{n}(2\,i\,\pi + a) \pm \sqrt{-1}\sin\frac{m}{n}(2\,i\,\pi + a) =$$

$$\cos\frac{m}{n}(2\,i\,\pi - a) \mp \sqrt{-1}\sin\frac{m}{n}(2\,i\,\pi - a)$$

which would evidently be an absurdity. It only shows that all the values of $[\cos a \pm \sqrt{-1}\sin a]^{\frac{m}{n}}$ are to be found in both formulæ by substituting for i the integers from 0 to $n-1$, and it does not follow that the contemporary values of i are the same. To find these quantities, suppose i and k to be contemporary,

$$\therefore\ \cos\frac{m}{n}(2\,i\,\pi + a) \pm \sqrt{-1}\sin\frac{m}{n}(2\,i\,\pi + a) =$$

$$\cos\frac{m}{n}(2\,k\,\pi - a) \mp \sqrt{-1}\sin\frac{m}{n}(2\,k\,\pi - a)$$

$\therefore$ by equating real and imaginary quantities and making $a = 0$

$$\cos\frac{m}{n}\,2\,i\,\pi = \cos\frac{m}{n}\,2\,k\,\pi$$

$$\sin\frac{m}{n}\,2\,i\,\pi = -\sin\frac{m}{n}\,2\,k\,\pi$$

$$\therefore\ \frac{m}{n}\,2\,i\,\pi = 2\,l\,\pi - \frac{m}{n}\,2\,k\,\pi,\ l \text{ being some integer,}$$

$$\therefore\ \frac{m\,(i+k)}{n} = l.$$

But $\frac{m}{n}$ is supposed to be in its lowest terms, therefore $i + k$ is a multiple of n, but $i + k$ is positive, and less than $2\,n$, $\therefore i + k = n$,

$\therefore\ k = n - i$

$$\therefore\ \cos\frac{m}{n}(2\,i\,\pi + a) \pm \sqrt{-1}\sin\frac{m}{n}(2\,i\,\pi + a) =$$

$$\cos\frac{m}{n}\{2\,(n-i)\,\pi - a\} \mp \sqrt{-1}\sin\frac{m}{n}\{2\,(n-i)\,\pi - a\}.$$

177. Cor. 2. Similarly it may be shown that all the values of $[\cos\alpha \pm \sqrt{-1}\sin\alpha]^{\frac{m}{n}}$ may be found in the formulæ

$$\cos\frac{m}{n}\{2(n-i)\pi+\alpha\} \pm \sqrt{-1}\sin\frac{m}{n}\{2(n-i)\pi+\alpha\},$$

$$\text{or, } \cos\frac{m}{n}(2i\pi-\alpha) \mp \sqrt{-1}\sin\frac{m}{n}(2i\pi-\alpha),$$

by substituting for i the integers from 0 to $n-1$.

178. Cor. 3. In the equation

$$[\cos\alpha\pm\sqrt{-1}\sin\alpha]^{\frac{m}{n}}=\cos\frac{m}{n}(2i\pi+\alpha)\pm\sqrt{-1}\sin\frac{m}{n}(2i\pi+\alpha)$$

let $\alpha = 0$

$$\therefore\ [1]^{\frac{m}{n}} = \cos\frac{m}{n}2i\pi\pm\sqrt{-1}\sin\frac{m}{n}2i\pi.$$

Now, since $\cos\frac{m}{n}2i\pi = \cos\frac{m}{n}2(n-i)\pi$

and $\sin\frac{m}{n}2i\pi = -\sin\frac{m}{n}2(n-i)\pi$

$$\therefore\ [1]^{\frac{m}{n}} = \cos\frac{m}{n}2(n-i)\pi \mp \sqrt{-1}\sin\frac{m}{n}2(n-i)\pi.$$

Hence it follows that the values of $[1]^{\frac{m}{n}}$ obtained by using the positive sign, and substituting $0, 1, 2 \ldots n-1$, for i are the same as when the negative sign is used, and $0, n-1, n-2, \ldots\ldots 3.\ 2.\ 1$ are put for i; and conversely.

179. Cor. 4. Let $\alpha = \pi$

$$\therefore[-1]^{\frac{m}{n}} = \cos\frac{m}{n}(2i+1)\pi\pm\sqrt{-1}\sin\frac{m}{n}(2i+1)\pi.$$

180. Cor. 5. Let $\alpha = \frac{\pi}{2}$

$$\therefore [\sqrt{-1}]^{\frac{m}{n}} = \cos\frac{m}{n}(4i+1)\frac{\pi}{2} \pm \sqrt{-1}\sin\frac{m}{n}.\ (4\,i+1)\,\frac{\pi}{2}.$$

181. PROP.

$$x^m - 1 = (x^2-1)\ (x^2 - 2x\cos\frac{2\pi}{m}+1)\,(x^2-2x\cos\frac{4\pi}{m}+1)\ldots.$$

to $\frac{m}{2}$ *factors, when m is even.*

$$x^m - 1 = (x-1)\ (x^2-2x\cos\frac{2\pi}{m}+1).\ (x^2-2x\cos\frac{4\pi}{m}+1)\ldots.$$

to $\frac{m+1}{2}$ *factors, when m is odd.*

For by making $a = 0$ in De Moivre's formula (art. 159) all the values of $[1]^{\frac{1}{m}}$ can be derived from the expression

$$\cos\frac{2\,i\,\pi}{m} + \sqrt{-1}\,\sin\frac{2\,i\,\pi}{m}\quad \text{(art. 178)}$$

by giving i the integral values from 0 to $m-1$.

Firstly, if m be even it is evident that $\frac{m}{2}$ will be one of the values of i. Hence, by the nature of equations, the roots of $x^m - 1 = 0$ are

$$
\begin{array}{ll}
i = 0, & \cos 0 + \sqrt{-1}\sin 0\ (= 1) \\
i = 1, & \cos\frac{2\pi}{m} + \sqrt{-1}\sin\frac{2\pi}{m} \\
i = 2, & \cos\frac{4\pi}{m} + \sqrt{-1}\sin\frac{4\pi}{m} \\
\ldots = \ldots & \ldots\ldots\ldots\ldots\ldots \\
i = \frac{m}{2} & \cos\pi + \sqrt{-1}\sin\pi\ (= -1) \\
\ldots = \ldots & \ldots\ldots\ldots\ldots\ldots
\end{array}
$$

$$i = m-2 \qquad \cos\frac{4\pi}{m} - \sqrt{-1}\sin\frac{4\pi}{m}$$

$$i = m-1 \qquad \cos\frac{2\pi}{m} - \sqrt{-1}\sin\frac{2\pi}{m}.$$

Hence, by collecting the pairs of factors and multiplying them together,

$$x^m - 1 = (x^2-1).\ (x^2 - 2\,x\cos\frac{2\pi}{m}+1)\ (x^2-2x\cos\frac{4\pi}{m}+1)\ldots$$

to $\frac{m}{2}$ factors, when m is even.

Secondly, when m is odd the process will be the same, excepting that $\frac{m}{2}$ will not be an integer, and therefore not one of the values of i, and the result will therefore be

$$x^m - 1 = (x-1)\ (x^2 - 2\,x\cos\frac{2\pi}{m}+1)\ (x^2-2x\cos\frac{4\pi}{m}+1)\ldots$$

to $\frac{m+1}{2}$ factors, when m is odd.

182. Prop.

$$x^m + 1 = (x^2 - 2\,x\cos\frac{\pi}{m} + 1)\ (x^2 - 2\,x\cos\frac{3\pi}{m} + 1)\ldots.$$

to $\frac{m}{2}$ *factors, when* m *is even.*

$$x^m + 1 = (x+1)\ (x^2 - 2x\cos\frac{\pi}{m} + 1)\ (x^2 - 2x\cos\frac{3\pi}{m} + 1)\ldots.$$

to $\frac{m+1}{2}$ *factors, when* m *is odd.*

For by making $\alpha = \pi$ in De Moivre's formula it appears that all the values of $[-1]^{\frac{1}{m}}$ can be derived from the expression

$$\cos\frac{(2i+1)\pi}{m} + \sqrt{-1}\sin\frac{(2i+1)\pi}{m}$$

by giving i the integral values from 0 to $m - 1$.

Hence, by the nature of equations, the roots of $x^m + 1 = 0$ are, when m is *even*,

$$\begin{array}{ll}
i = 0 & \cos\frac{\pi}{m} + \sqrt{-1}\sin\frac{\pi}{m} \\
i = 1 & \cos\frac{3\pi}{m} + \sqrt{-1}\sin\frac{3\pi}{m} \\
i = 2 & \cos\frac{5\pi}{m} + \sqrt{-1}\sin\frac{5\pi}{m} \\
\ldots = \ldots & \ldots\ldots\ldots\ldots\ldots \\
i = m-3 & \cos\frac{5\pi}{m} - \sqrt{-1}\sin\frac{5\pi}{m} \\
i = m-2 & \cos\frac{3\pi}{m} - \sqrt{-1}\sin\frac{3\pi}{m} \\
i = m-1 & \cos\frac{\pi}{m} - \sqrt{-1}\sin\frac{\pi}{m}.
\end{array}$$

Hence, by collecting the pairs of factors and multiplying them together,

$$x^m + 1 = (x^2 - 2x\cos\frac{\pi}{m} + 1)(x^2 - 2x\cos\frac{3\pi}{m} + 1)\ldots\ldots$$

to $\frac{m}{2}$ factors, when m is even.

When m is *odd*, $m - 1$ is even, and one of the values of i will be $\frac{m-1}{2}$. The root of $x^m - 1 = 0$ corresponding to this value of i will be

$$\cos\pi + \sqrt{-1}\sin\pi \;(= -1)$$

and the factor will therefore be $x + 1$

$\therefore\ x^m+1=(x+1)\left(x^2-2x\cos\frac{\pi}{m}+1\right)\left(x^2-2x\cos\frac{3\pi}{m}+1\right)\ldots.$

to $\frac{m+1}{2}$ factors, when m is odd.

183. Prop. $x^{2m}-2x^m\cos\alpha+1=\left(x^2-2x\cos\frac{\alpha}{m}+1\right)$

$\times\ \left(x^2-2x\cos\frac{2\pi+\alpha}{m}+1\right)\left(x^2-2x\cos\frac{4\pi+\alpha}{m}+1\right)$

$\times\ \ldots\ldots$

$\times\ \left(x^2-2x\cos\frac{2(m-1)\pi+\alpha}{m}+1\right).$

For, by solving the quadratic equation,

$$x^{2m}-2x^m\cos\alpha+1=0$$

it appears that

$$x^m=\cos\alpha\pm\sqrt{-1}\sin\alpha$$

$$x=[\cos\alpha\pm\sqrt{-1}\sin\alpha]^{\frac{1}{m}}$$

$$=\cos\frac{2i\pi+\alpha}{m}\pm\sqrt{-1}\sin\frac{2i\pi+\alpha}{m}$$

i being any integer from 0 to $n-1$.

Therefore the values of x are

$i=0,\quad \cos\frac{\alpha}{m}\pm\sqrt{-1}\sin\frac{\alpha}{m}$

$i=1,\quad \cos\frac{2\pi+\alpha}{m}\pm\sqrt{-1}\sin\frac{2\pi+\alpha}{m}$

$i=2,\quad \cos\frac{4\pi+\alpha}{m}\pm\sqrt{-1}\sin\frac{4\pi+\alpha}{m}$

$\ldots=\ldots\quad \ldots\ldots\ldots\ldots\ldots\ldots$

$i=m-1,\cos\frac{2(m-1)\pi+\alpha}{m}\pm\sqrt{-1}\sin\frac{2(m-1)\pi+\alpha}{m}.$

Hence, by reducing the pairs of simple into quadratic factors,

$$x^{2m} - 2x^m \cos a + 1 = \left(x^2 - 2\,x \cos \frac{a}{m} + 1\right)$$

$$\times \left(x^2 - 2\,x \cos \frac{2\,\pi + a}{m} + 1\right) \left(x^2 - 2\,x \cos \frac{4\,\pi + a}{m} + 1\right)$$

$$\times \ldots$$

$$\times \left(x^2 - 2\,x \cos \frac{2\,(m-1)\,\pi + a}{m} + 1\right).$$

184. Cor. $\dfrac{2\,(m-i)\,\pi + a}{m} = 2\,\pi - \dfrac{2\,i\,\pi - a}{m}$

$$\therefore \cos \frac{2\,(m-i)\,\pi + a}{m} = \cos \frac{2\,i\,\pi - a}{m}$$

$$\therefore\ x^{2m} \quad 2\,x^m \cos a + 1 = \left(x^2 - 2\,x \cos \frac{a}{m} + 1\right)$$

$$\times \left(x^2 - 2\,x \cos \frac{2\,\pi + a}{m} + 1\right) \left(x^2 - 2\,x \cos \frac{2\,\pi - a}{m} + 1\right)$$

$$\times \left(x^2 - 2\,x \cos \frac{4\,\pi + a}{m} + 1\right) \left(x^2 - 2\,x \cos \frac{4\,\pi - a}{m} + 1\right)$$

$\times$ ..... to $\dfrac{m}{2}$ factors.

185. Prop.

$$Cos\ \frac{m}{n}\,(2\,i\,\pi \pm a) = (cos\ a)^{\frac{m}{n}} \left(\mathrm{T} cos\ \frac{m}{n}\,2i\pi \mp \mathrm{T'} sin\ \frac{m}{n}\,2\,i\,\pi\right)$$

$$sin\ \frac{m}{n}\,(2\,i\,\pi \pm a) = (cos\ a)^{\frac{m}{n}} \left(\mathrm{T} sin\ \frac{m}{n}\,2i\pi \pm \mathrm{T'} cos\ \frac{m}{n}\,2\,i\,\pi\right)$$

a being not $> \dfrac{\pi}{2}$, $\dfrac{m}{n}$ a fraction in its lowest terms,

i any integer from 0 to $n - 1$,

$$\mathrm{T} = 1 - \frac{m}{n}\left(\frac{m}{n} - 1\right)\frac{(tan\ a)^2}{1.\ 2} + \ldots \textit{and}$$

$$\mathrm{T'} = \frac{m}{n}\,tan\ a - \frac{m}{n}\cdot\left(\frac{m}{n} - 1\right)\cdot\left(\frac{m}{n} - 2\right)\cdot\frac{(tan\ a)^3}{1.\ 2.\ 3} + \ldots$$

$$\therefore\ x^m+1=(x+1)\,(x^2-2x\cos\frac{\pi}{m}+1)\,(x^2-2x\cos\frac{3\pi}{m}+1)\ldots.$$

to $\dfrac{m+1}{2}$ factors, when m is odd.

183. Prop. $x^{2m}-2\,x^m\cos a+1=(x^2-2\,x\cos\frac{a}{m}+1)$

$\times\ (x^2-2\,x\cos\dfrac{2\,\pi+a}{m}+1)\ (x^2-2\,x\cos\dfrac{4\,\pi+a}{m}+1)$

$\times\ \ldots\ldots$

$\times\ (x^2-2\,x\cos\dfrac{2\,(m-1)\,\pi+a}{m}+1).$

For, by solving the quadratic equation,

$$x^{2m}-2\,x^m\cos a+1=0$$

it appears that

$$x^m=\cos a\pm\sqrt{-1}\sin a$$

$$x=[\cos a\pm\sqrt{-1}\sin a]^{\frac{1}{m}}$$

$$=\cos\frac{2\,i\,\pi+a}{m}\pm\sqrt{-1}\sin\frac{2\,i\,\pi+a}{m}$$

i being any integer from 0 to $n-1$.

Therefore the values of x are

$i=0,\quad \cos\dfrac{a}{m}\pm\sqrt{-1}\sin\dfrac{a}{m}$

$i=1,\quad \cos\dfrac{2\,\pi+a}{m}\pm\sqrt{-1}\sin\dfrac{2\,\pi+a}{m}$

$i=2,\quad \cos\dfrac{4\,\pi+a}{m}\pm\sqrt{-1}\sin\dfrac{4\,\pi+a}{m}$

$\ldots=\ldots\quad \ldots\ldots\ldots\ldots\ldots\ldots$

$i=m-1,\ \cos\dfrac{2\,(m-1)\,\pi+a}{m}\pm\sqrt{-1}\sin\dfrac{2\,(m-1)\pi+a}{m}.$

Hence, by reducing the pairs of simple into quadratic factors,

$$x^{2m} - 2x^m \cos a + 1 = (x^2 - 2\,x \cos \frac{a}{m} + 1)$$

$$\times\ (x^2 - 2\,x \cos \frac{2\,\pi + a}{m} + 1)\ (x^2 - 2\,x \cos \frac{4\,\pi + a}{m} + 1)$$

$$\times \ldots$$

$$\times\ (x^2 - 2\,x \cos \frac{2\,(m-1)\,\pi + a}{m} + 1).$$

184. Cor. $\dfrac{2\,(m-i)\,\pi + a}{m} = 2\,\pi - \dfrac{2\,i\,\pi - a}{m}$

$$\therefore\ \cos \frac{2\,(m-i)\,\pi + a}{m} = \cos \frac{2\,i\,\pi - a}{m}$$

$$\therefore\ x^{2m} \quad 2\,x^m \cos a + 1 = (x^2 - 2\,x \cos \frac{a}{m} + 1)$$

$$\times\ (x^2 - 2\,x \cos \frac{2\,\pi + a}{m} + 1)\ (x^2 - 2\,x \cos \frac{2\,\pi - a}{m} + 1)$$

$$\times\ (x^2 - 2\,x \cos \frac{4\,\pi + a}{m} + 1)\ (x^2 - 2\,x \cos \frac{4\,\pi - a}{m} + 1)$$

$$\times\ \ldots\ldots \text{ to } \frac{m}{2} \text{ factors.}$$

185. Prop.

$$Cos\,\frac{m}{n}\,(2\,i\,\pi \pm a) = (cos\,a)^{\frac{m}{n}}\,(\mathrm{T}cos\,\frac{m}{n}\,2i\pi \mp \mathrm{T}'sin\,\frac{m}{n}\,2\,i\,\pi)$$

$$sin\,\frac{m}{n}\,(2\,i\,\pi \pm a) = (cos\,a)^{\frac{m}{n}}\,(\mathrm{T}sin\,\frac{m}{n}\,2i\pi \pm \mathrm{T}'cos\,\frac{m}{n}\,2\,i\,\pi)$$

a *being not* $> \frac{\pi}{2}$, $\frac{m}{n}$ *a fraction in its lowest terms,*

i *any integer from* 0 *to* $n - 1$,

$$\mathrm{T} = 1 - \frac{m}{n}\left(\frac{m}{n} - 1\right)\frac{(tan\,a)^2}{1.\ 2} + \ldots \textit{and}$$

$$\mathrm{T}' = \frac{m}{n}\,tan\,a - \frac{m}{n}\cdot\left(\frac{m}{n} - 1\right)\cdot\left(\frac{m}{n} - 2\right)\cdot\frac{(tan\,a)^3}{1.\ 2.\ 3} + \ldots$$

For, firstly,

$$\cos\frac{m}{n}(2i\pi+a)+\sqrt{-1}\sin\frac{m}{n}(2i\pi+a)=[\cos a+\sqrt{-1}\sin a]^{\frac{m}{n}}$$

$$=[\cos a]^{\frac{m}{n}}(\mathrm{T}+\sqrt{-1}\,\mathrm{T}')=(\cos a)^{\frac{m}{n}}[1]^{\frac{m}{n}}(\mathrm{T}+\sqrt{-1}\,\mathrm{T}')$$

$$=(\cos a)^{\frac{m}{n}}\left(\cos\frac{m}{n}2k\pi+\sqrt{-1}\sin\frac{m}{n}2k\pi\right)(\mathrm{T}+\sqrt{-1}\,\mathrm{T}')$$

k being some integer from 0 to $n-1$. (art. 178.)

$\therefore$ by multiplying and equating real and imaginary quantities

$$\cos\frac{m}{n}(2i\pi+a)=(\cos a)^{\frac{m}{n}}\left(\mathrm{T}\cos\frac{m}{n}2k\pi-\mathrm{T}'\sin\frac{m}{n}2k\pi\right)$$

$$\sin\frac{m}{n}(2i\pi+a)=(\cos a)^{\frac{m}{n}}\left(\mathrm{T}\sin\frac{m}{n}2k\pi+\mathrm{T}'\cos\frac{m}{n}2k\pi\right).$$

Let $a=0$ $\therefore$ $(\cos a)^{\frac{m}{n}}=1$, $\mathrm{T}=1$, and $\mathrm{T}'=0$,

$$\therefore \cos\frac{m}{n}2i\pi=\cos\frac{m}{n}2k\pi$$

$$\text{and } \sin\frac{m}{n}2i\pi=\sin\frac{m}{n}2k\pi$$

$\therefore \frac{m}{n}2i\pi=2l\pi+\frac{m}{n}2k\pi$, l being some integer,

$$\therefore \frac{m(i-k)}{n}=l,$$

but $\frac{m}{n}$ is in its lowest terms, $\therefore$ $i-k$ is a multiple of n, but $i-k$ cannot be less than $-(n-1)$, nor greater than $n-1$, $\therefore$ $i-k=0$ $\therefore$ $i=k$

$$\therefore \cos\frac{m}{n}(2i\pi+a)=(\cos a)^{\frac{m}{n}}\left(\mathrm{T}\cos\frac{m}{n}2i\pi-\mathrm{T}'\sin\frac{m}{n}2i\pi\right)$$

$$\text{and } \sin\frac{m}{n}(2i\pi+a)=(\cos a)^{\frac{m}{n}}\left(\mathrm{T}\sin\frac{m}{n}2i\pi+\mathrm{T}'\cos\frac{m}{n}2i\pi\right).$$

Secondly. If a be negative, $\cos a$ and T remain unchanged, but T′ becomes $-\mathrm{T}'$,

$\therefore \cos\frac{m}{n}(2i\,\pi\pm\alpha)=(\cos\alpha)^{\frac{m}{n}}\left(\mathrm{T}\cos\frac{m}{n}2i\,\pi\mp\mathrm{T}'\sin\frac{m}{n}2i\,\pi\right)$

and $\sin\frac{m}{n}(2i\,\pi\pm\alpha)=(\cos\alpha)^{\frac{m}{n}}\left(\mathrm{T}\sin\frac{m}{n}2i\,\pi\pm\mathrm{T}'\cos\frac{m}{n}2i\,\pi\right)$.

186. Cor.

$$\cos\frac{m}{n}\{(2i+1)\,\pi+\alpha\}+\sqrt{-1}\sin\frac{m}{n}\{(2i+1)\,\pi+\alpha\}$$
$$=\cos\frac{m}{n}\{2i\,\pi+(\pi+\alpha)\}+\sqrt{-1}\sin\frac{m}{n}\{2i\,\pi+(\pi+\alpha)\}$$
$$=[\cos(\pi+\alpha)+\sqrt{-1}\sin(\pi+\alpha)]^{\frac{m}{n}}$$
$$=[-\cos\alpha-\sqrt{-1}\sin\alpha]^{\frac{m}{n}}$$
$$=[-1]^{\frac{m}{n}}(\mathrm{T}+\sqrt{-1}\,\mathrm{T}')$$

$\therefore$ by a process similar to that in the proposition,

$$\cos\frac{m}{n}\{(2i+1)\pi\pm\alpha\}=$$
$$(\cos\alpha)^{\frac{m}{n}}\left\{\mathrm{T}\cos\frac{m}{n}(2i+1)\,\pi\mp\mathrm{T}'\sin\frac{m}{n}(2i+1)\,\pi\right\}$$
$$\sin\frac{m}{n}\{(2i+1)\pi\pm\alpha\}=$$
$$(\cos\alpha)^{\frac{m}{n}}\left\{\mathrm{T}\sin\frac{m}{n}(2i+1)\,\pi\pm\mathrm{T}'\cos\frac{m}{n}(2i+1)\,\pi\right\}.$$

187. Prob. *To find what values of i will cause $\cos\frac{m}{n}2i\,\pi$ to vanish.*

If $\cos\frac{m}{n}2i\,\pi=\cos\frac{m}{n}4i\frac{\pi}{2}=0$

$$\frac{m}{n}4i=\text{an odd integer.}$$

But since $\frac{m}{n}$ is in its lowest terms, m is prime to n

$\therefore$ m and $\frac{4i}{n}$ are both odd integers.

$$\text{But } i < n$$
$$\therefore \frac{4i}{n} < 4$$
$$\therefore \frac{4i}{n} = 1, \text{ or } = 3$$
$$\therefore i = \frac{n}{4}, \text{ or } = \frac{3n}{4}.$$

And since these are integers n is of the form $4n'$. In the *first* of these cases $\sin \frac{m}{n} 2i\pi = \sin m\frac{\pi}{2} = \pm 1$; $\pm$ according as m is of the form $4m' + 1$, or $4m' + 3$. In the *second* case $\sin \frac{m}{n} 2i\pi = \sin 3m\frac{\pi}{2} = \mp 1$; $\mp$ according as m is of the form $4m' + 1$, or $4m' + 3$. Hence whenever m is odd and n is divisible by 4,

$$\left.\begin{aligned} \cos \frac{m}{n}\left(\frac{n}{2}\pi + \alpha\right) &= \mp (\cos \alpha)^{\frac{m}{n}} \mathrm{T}' \\ \sin \frac{m}{n}\left(\frac{n}{2}\pi + \alpha\right) &= \pm (\cos \alpha)^{\frac{m}{n}} \mathrm{T} \end{aligned}\right\}$$

$$\left.\begin{aligned} \cos \frac{m}{n}\left(\frac{3n}{2}\pi + \alpha\right) &= \pm (\cos \alpha)^{\frac{m}{n}} \mathrm{T}' \\ \sin \frac{m}{n}\left(\frac{3n}{2}\pi + \alpha\right) &= \mp (\cos \alpha)^{\frac{m}{n}} \mathrm{T} \end{aligned}\right\}.$$

188. PROB. *To find what values of i will cause $\sin \frac{m}{n}2i\pi$ to vanish.*

$$\text{If } \sin \frac{m}{n} 2i\pi = 0$$
$$\frac{m}{n} 2i = \text{an integer},$$

but m is prime to n,

$$\therefore \frac{2i}{n} = \text{an integer}$$

$$\text{but } i < n$$

$$\therefore \frac{2i}{n} < 2$$

$$\therefore \frac{2i}{n} = 0, \text{ or } 1$$

$$\therefore i = 0, \text{ or } \frac{n}{2}.$$

In the *first* case the formulæ become

$$\left.\begin{aligned} \cos \frac{m}{n}\, a &= (\cos a)^{\frac{m}{n}}\, T \\ \sin \frac{m}{n}\, a &= (\cos a)^{\frac{m}{n}}\, T'. \end{aligned}\right\} \text{(See art. 162.)}$$

These are the common series which for a fraction are therefore confined to the real value of $[\cos a]^{\frac{m}{n}}$ and to a less than $\frac{\pi}{2}$. In the *second* case n must be even, and therefore m odd, $\cos \frac{m}{n} 2i\, \pi = \cos m\, \pi = -1$; and the formulæ become

$$\cos \frac{m}{n} (n\, \pi + a) = -(\cos a)^{\frac{m}{n}}\, T$$

$$\sin \frac{m}{n} (n\, \pi + a) = -(\cos a)^{\frac{m}{n}}\, T'.$$

189. PROB. *To find what values of i will cause $\sin \frac{m}{n} (2i + 1)\, \pi$ to vanish.*

$$\text{If } \sin \frac{m}{n} (2i + 1)\, \pi = 0$$

$$\frac{m}{n} (2i + 1) = \text{an integer}$$

$$\therefore \frac{(2i+1)}{n} = \text{an integer,}$$

$$\text{but } i < n$$

$$\therefore \frac{(2i+1)}{n} < 2$$

$$\therefore \frac{(2i+1)}{n} = 0, \text{ or } = 1$$

$$\therefore i = -\tfrac{1}{2}, \text{ or } = \frac{n-1}{2}.$$

The first of these is inadmissible, because i must be a positive integer; the second shows that n *must be odd.* In this case $\cos \frac{m}{n}(2i+1)\pi = \cos m\pi = \pm 1$, according as m is even or odd. The formulæ become

$$\cos \frac{m}{n}(n\pi + a) = \pm (\cos a)^{\frac{m}{n}}\, \mathrm{T}$$

$$\cos \frac{m}{n}(n\pi + a) = \pm (\cos a)^{\frac{m}{n}}\, \mathrm{T}'.$$

190. PROB. *To find what values of i will cause $\cos \frac{m}{n}(2i+1)\pi$ to vanish.*

$$\text{If } \cos \frac{m}{n}(2i+1)\pi = 0$$

$$\frac{2m(2i+1)}{n} = \text{an odd integer,}$$

but m is prime to n,

$$\therefore m \text{ and } \frac{2(2i+1)}{n} \text{ are both odd integers.}$$

$$\text{But } i \text{ is } < n$$

$$\therefore \frac{2(2i+1)}{n} < 4$$

$$\therefore \frac{2(2i+1)}{n} = 1, \text{ or } = 3$$

$$\therefore i = \frac{n-2}{4}, \text{ or } = \frac{3n-2}{4},$$

and these are integers, therefore n of the form $4\,n' + 2$. In the *first* case $\sin \frac{m}{n}(2i+1)\pi = \sin m\frac{\pi}{2} = \pm 1$; $\pm$ according as m is of the form $4\,m' + 1$, or $4\,m' + 3$, and the formulæ become

$$\cos \frac{m}{n}\left(\frac{n}{2}\pi + a\right) = \mp (\cos a)^{\frac{m}{n}}\, T'$$

$$\sin \frac{m}{n}\left(\frac{n}{2}\pi + a\right) = \pm (\cos a)^{\frac{m}{n}}\, T.$$

In the *second* case $\sin \frac{m}{n}(2i+1)\pi = \sin 3\,m\frac{\pi}{2} = \mp 1$; $\mp$ according as m is of the form $4\,m' + 1$, or $4\,m' + 3$, and

$$\cos \frac{m}{n}\left(3\frac{n}{2}\pi + a\right) = \pm (\cos a)^{\frac{m}{n}}\, T'$$

$$\sin \frac{m}{n}\left(3\frac{n}{2}\pi + a\right) = \pm (\cos a)^{\frac{m}{n}}\, T.$$

191. PROP. $2^{m-1}(\cos a)^m = \cos m\,a + m.\cos(m-2)a + \frac{m.(m-1)}{1.\ 2}\cos(m-4)\,a + \ldots$

$$+ \tfrac{1}{2}.\frac{m\ (m-1)\ \ldots\ \left(\frac{m}{2}+1\right)}{1.\ 2.\ 3\ \ldots\ \frac{m}{2}}$$

m being a positive integer, and the last term being added only when m is an even number.

For let $2\cos a = x + \frac{1}{x}$

$$\therefore 2^m (\cos a)^m = x^m + m\,x^{m-2} + \frac{m.\,m-1}{1.\ 2}x^{m-4} + \ldots$$

But $x^m = \cos m\,a + \sqrt{-1}\sin m\,a$

$x^{m-2} = \cos(m-2)\,a + \sqrt{-1}\sin(m-2)\,a$

$\ldots\ldots = \ldots\ldots$

$$\therefore\ 2^m(\cos a)^m = \cos m a + m.\cos(m-2)\,a + \ldots.$$
$$+ \sqrt{-1}\ \{\sin m\,a + m\sin(m-2)\,a + \ldots\}.$$

But since m is a positive integer, $2^m (\cos a)^m$ cannot involve any imaginary quantity, therefore the imaginary part of the equation must vanish,

$$\therefore\ 2^m (\cos a)^m = \cos m a + m \cos (m - 2) a + \ldots$$

This series will terminate after $m + 1$ terms; the last term will therefore be

$$\cos (m - 2m) a = \cos (- m a) = \cos m a\ ;$$

the last but one,

$$m \cos \{m - 2 (m - 1)\} a = m \cos (2 - m) a = \cos (m - 2) a,$$

$$\therefore\ 2^m (\cos a)^m = 2 \cos m a + 2m \cos (m - 2) a + \ldots$$

Now if m be odd, $m + 1$ will be even, and there will be exactly $\frac{m+1}{2}$ pairs of equal terms. But if m be even, $m + 1$ will be odd; there will be $\frac{m}{2}$ pairs of equal terms, and a single middle term. This will be the $\left(\frac{m}{2} + 1\right)^{\text{th}}$ term, and therefore

$$= \frac{m.\ (m - 1) \ldots \ldots \left(m - (\frac{m}{2} - 1)\right)}{1.\ 2.\ 3 \ldots \ldots \frac{m}{2}} \cos \left(m - 2\frac{m}{2}\right) a$$

$$= \frac{m.\ (m - 1) \ldots \ldots \left(\frac{m}{2} + 1\right)}{1.\ 2.\ 3 \ldots \ldots \frac{m}{2}}$$

$$\therefore\ 2^m (\cos a)^m = 2 \cos m a + 2\, m \cos (m - 2) a + \ldots$$

$$+ \frac{m.\ (m-1) \ldots \left(\frac{m}{2} + 1\right)}{1.\ 2.\ 3 \ldots \frac{m}{2}}$$

$$\therefore\ 2^{m-1} (\cos a)^m = \cos m a + m \cos (m - 2) a + \ldots$$

$$+ \tfrac{1}{2}. \frac{m.\ (m-1) \ldots \left(\frac{m}{2} + 1\right)}{1.\ 2.\ 3 \ldots \frac{m}{2}}.$$

192. COR. $$\frac{m.(m-1)(m-2)\ldots\left(\frac{m}{2}+2\right).\left(\frac{m}{2}+1\right)}{\frac{m}{2}.\left(\frac{m}{2}-1\right).\ldots\ldots\quad 3.\ 2.\ 1}$$

$$= 2^{\frac{m}{2}}.\frac{m.(m-1).(m-2)\ldots\left(\frac{m}{2}+2\right)\left(\frac{m}{2}+1\right)}{m.(m-2).(m-4)\ldots\quad 6.\ 4.\ 2}$$

$$= 2^{\frac{m}{2}}.\frac{m.(m-1).(m-2)\ldots\left(\frac{m}{2}+2\right)\left(\frac{m}{2}+4\right)}{m.(m-1).(m-2)\ldots\quad 3.\ 2.\ 1}\times 1.\ 3.\ 5..(m-1)$$

$$= 2^{\frac{m}{2}}.\frac{m.(-1).\ \ldots\ \left(\frac{m}{2}+2\right).\left(\frac{m}{2}+1\right)}{m.(m-1).\ \ldots\ \left(\frac{m}{2}+2\right).\left(\frac{m}{2}+1\right)}\times\frac{1.\ 3.\ 5.\ldots(m-1)}{\frac{m}{2}.\left(\frac{m}{2}-1\right)\ldots 2.1}$$

$$= 2^{\frac{m}{2}}.\frac{1.\ 3.\ 5\ \ldots\ldots\ m.(m-1)}{1.\ 2.\ 3\ \ldots\ldots\ \frac{m}{2}}$$

$$\therefore\ 2^{m-1}(\cos a)^m = \cos ma + m.\cos(m-2)a + \ldots$$
$$+\ 2^{\frac{m}{2}-1}.\frac{1.\ 3.\ 5\ \ldots\ m.(m-1)}{1.\ 2.\ 3\ \ldots\ \frac{m}{2}}.$$

193. PROP. *When m is a positive integer of the form*

$$\left.\begin{array}{l}4m',\\ 4m'+2\end{array}\right\}2^{m-1}(\sin a)^m = \pm\left\{\cos ma - m\cos(m-2)a\right.$$

$$\left.+\frac{m.(m-1)}{1.\ 2}\cos(m-4)a - \ldots \text{ to } \frac{m}{2} \text{ terms}\right\} + \tfrac{1}{2}\,\frac{m.(m-1)\ldots\left(\frac{m}{n}+1\right)}{1.\ 2\ \ldots\ \frac{m}{2}}$$

$$\left.\begin{array}{l}4m'+1\\ 4m'+3\end{array}\right\}2^{m-1}(\sin a)^m = \pm\left\{\sin ma - m\sin(m-2)a\right.$$

$$\left.+\ \frac{m.(m-1)}{1.\ 2}\sin(m-4)a - \ldots \text{ to } \frac{m+1}{2} \text{ terms.}\right\}$$

Let $2\sqrt{-1}\sin\alpha = x - \dfrac{1}{x}$

$$\therefore\ 2^m(\sqrt{-1})^m(\sin\alpha)^m = x^m - mx^{m-2} + \frac{m.(m-1)}{1.\ 2}x^{m-4} - \ldots$$

$$\text{But } x^m = \cos m\alpha + \sqrt{-1}\sin m\alpha$$
$$x^{m-2} = \cos(m-2)\alpha + \sqrt{-1}\sin(m-2)\alpha$$
$$\ldots = \ldots.$$
$$2^m(\sqrt{\ 1})^m(\sin\alpha)^m = \cos m\alpha - m\cos(m-2)\alpha + \ldots$$
$$+\sqrt{-1}\{\sin m\alpha - m\sin(m-2)\alpha + \ldots\}.$$

Now when m is of the form

$$4m',\quad (\sqrt{-1})^m = (\sqrt{-1})^{4m'} = (-1)^{2m'} = 1$$
$$4m' + 2\ (\sqrt{-1})^m = (\sqrt{-1})^{4m'}(\sqrt{-1})^2 = -1$$

therefore, in these cases, by equating real quantities,

$$2^m(\sin\alpha)^m = \pm\{\cos m\alpha - m\cos(m-2)\alpha + \ldots \text{to } (m+1)\text{terms}\}$$
$$= \pm\{2\cos m\alpha - 2m\cos(m-2)\alpha + \ldots \text{to } \frac{m}{2} \text{ terms}$$
$$\pm\frac{m.(m-1)\ldots\left(\frac{m}{2}+1\right)}{1.\ 2.\ 3\ \ldots\ \frac{m}{2}}\}$$
$$\therefore\ 2^{m-1}(\sin\alpha)^m = \pm\ \{\cos m\alpha - m\cos(m-2)\alpha + \ldots\}$$
$$+\frac{1}{2}\cdot\frac{m.(m-1)\ldots\left(\frac{m}{2}+1\right)}{1.\ 2.\ 3\ \ldots\ \frac{m}{2}}.$$

When m is of the form

$$4m' + 1,\ (\sqrt{-1})^m = (\sqrt{-1})^{4m'}\sqrt{-1} = \sqrt{-1}$$
$$4m' + 3,\ (\sqrt{-1})^m = (\sqrt{-1})^{4m'}(\sqrt{-1})^3 = -(\sqrt{-1}$$

therefore, by equating imaginary quantities, and dividing by $\sqrt{-1}$

$$2^m(\sin\alpha)^m = \pm\{\sin m\alpha - m\sin(m-2\alpha + \ldots \text{to } m+1 \text{ terms}\}$$
$$= \pm\{2\sin m\alpha - 2m\sin(m-2)\alpha + \ldots \text{to } \frac{m+1}{2} \text{ terms}\}$$
$$\therefore 2^{m-1}(\sin\alpha)^m = \pm\{\sin m\alpha - m\sin(m-2)\alpha + \ldots \text{to } \frac{m+1}{2} \text{terms}\}$$

194. *If* $C_\alpha = \cos\frac{m}{n}\alpha + \frac{m}{n}\cos\left(\frac{m}{n}-2\right)\alpha + \ldots$

$$S_\alpha = \sin\frac{m}{n}\alpha + \frac{m}{n}\sin\left(\frac{m}{n}-2\right)\alpha + \ldots$$

α being less than $\frac{\pi}{2}$, and i any number from 0 to $n-1$, then

$$(2\cos\alpha)^{\frac{m}{n}} = \frac{C_{2i\pi+\alpha}}{\cos\frac{m}{n}2i\pi} = \frac{S_{2i\pi+\alpha}}{\sin\frac{m}{n}2i\pi}.$$

For, $[2\cos\alpha]^{\frac{m}{n}} = (2\cos\alpha)^{\frac{m}{n}}(\cos\frac{m}{n}2i\pi + \sqrt{-1}\sin\frac{m}{n}2i\pi)$

i being any integer.

Let $2\cos\alpha = x + \frac{1}{x}$

and $\therefore x = \cos\alpha + \sqrt{-1}\sin\alpha$

$$\therefore [2\cos\alpha]^{\frac{m}{n}} = [x]^{\frac{m}{n}} + \frac{m}{n}[x]^{\frac{m-2}{n}} + \ldots$$

$$= \cos\frac{m}{n}(2k\pi+\alpha) + \sqrt{-1}\sin\frac{m}{n}(2k\pi+\alpha)$$

$$+ \frac{m}{n}\cos\left(\frac{m}{n}-2\right)(2k\pi+\alpha) + \sqrt{-1}\sin\left(\frac{m}{n}-2\right)(2k\pi+\alpha)$$

$$+ \ldots$$

$$= C_{2k\pi+\alpha} + \sqrt{-1}\,S_{2k\pi+\alpha}$$

$$\therefore (2\cos\alpha)^{\frac{m}{n}}\left(\cos\frac{m}{n}2i\pi + \sqrt{-1}\sin\frac{m}{n}2i\pi\right) = C_{2k\pi+\alpha} + \sqrt{-1}\,S_{2k\pi+\alpha}$$

$\therefore$ by equating real and imaginary quantities

$$(2\cos\alpha)^{\frac{m}{n}}\cos\frac{m}{n}2i\pi = C_{2k\pi+\alpha}$$

$$(2\cos\alpha)^{\frac{m}{n}}\sin\frac{m}{n}2i\pi = S_{2k\pi+\alpha}.$$

By making $\alpha = 0$

$$2^{\frac{m}{n}}\cos\frac{m}{n}2i\pi = C_{2k\pi}$$

$$= \cos\frac{m}{n}2k\pi + \frac{m}{n}\cos\left(\frac{m}{n}-2\right)2k\pi + .$$

$$= \cos \frac{m}{n} 2 k \pi + \frac{m}{n} \cos \frac{m}{n} 2 k \pi + \ldots$$

$$= \cos \frac{m}{n} 2 k \pi \; (1 + 1)^{\frac{m}{n}}$$

$$= 2^{\frac{m}{n}} \cos \frac{m}{n} 2 k \pi$$

$$\therefore \cos \frac{m}{n} 2 i \pi = \cos \frac{m}{n} 2 k \pi,$$

and by restricting the values of i and k to $0, n - 1$, and the included integers, as in art. 185, $i = k$,

$$\therefore (2 \cos a)^{\frac{m}{n}} = \frac{C_{2 i \pi + a}}{\cos \frac{m}{n} 2 i \pi}$$

$$\therefore (2 \cos a)^{\frac{m}{n}} = \frac{S_{2 i \pi + a}}{\sin \frac{m}{n} 2 i \pi}.$$

195. Cor. By division

$$\frac{S_{2 i \pi + a}}{C_{2 i \pi + a}} = \frac{\sin \frac{m}{n} 2 i \pi}{\cos \frac{m}{n} 2 i \pi} = \tan \frac{m}{n} 2 i \pi.$$

196. Prop. $(2 \cos a)^{\frac{m}{n}} = \dfrac{C_{(2 i + 1) \pi + a}}{\cos \frac{m}{n} (2i + 1) \pi} = \dfrac{S_{(2 i + 1) \pi + a}}{\sin \frac{m}{n} (2i + 1) \pi}$

the same notation being used as in art. 194.

For, by art. 179,

$$[2 \cos (\pi + a)]^{\frac{m}{n}} = (2 \cos a)^{\frac{m}{n}} \; [-1]^{\frac{m}{n}}$$

$$= (2 \cos a)^{\frac{m}{n}} \left\{\cos \frac{m}{n} (2i + 1) \pi + \sqrt{-1} \sin \frac{m}{n} (2i + 1) \pi\right\}$$

Let $2\cos(\pi+a) = x + \frac{1}{x}$, or $x = \cos(\pi+a) + \sqrt{-1}\sin(\pi+a)$

$$[2\cos(\pi+a)]^{\frac{m}{n}} = x^{\frac{m}{n}} + \frac{m}{n}.\, x^{\frac{m}{n}-2} + \frac{m}{n}\frac{\left(\frac{m}{n}-1\right)}{2} x^{\frac{m}{n}-4} + \ldots$$

$$= \cos\frac{m}{n}\{(2k+1)\pi+a\} + \sqrt{-1}\sin\frac{m}{n}\{(2k+1)\pi+a\}$$

$$+ \frac{m}{n}\{\cos\left(\frac{m}{n}-2\right)\{(2k+1)\pi+a\}$$

$$+ \sqrt{-1}\sin\left(\frac{m}{n}-2\right)\{(2k+1)\pi+a\}$$

$$+ \ldots$$

$$= C_{(2k+1)\pi+a} + \sqrt{-1}\, S_{(2k+1)\pi+a}.$$

Therefore, by operating as in the art. 194.

$$(2\cos a)^{\frac{m}{n}} = \frac{C_{(2i+1)\pi+a}}{\cos\frac{m}{n}(2i+1)\pi}$$

$$\text{and } (2\cos a)^{\frac{m}{n}} = \frac{S_{(2i+1)\pi+a}}{\sin\frac{m}{n}(2i+1)\pi}.$$

197. Cor.

$$\frac{S_{(2i+1)\pi+a}}{C_{(2i+1)\pi+a}} = \frac{\sin\frac{m}{n}(2i+1)\pi}{\cos\frac{m}{n}(2i+1)\pi} = \tan\frac{m}{n}(2i+1)\pi.$$

198. Prob. *To find what value of i will cause* $C_{2i\pi+a}$ *to vanish.*

Because $(2\cos a)^{\frac{m}{n}} = \dfrac{C_{2i\pi+a}}{\cos\frac{m}{n}2i\pi}$

If, then, $C_{2i\pi+a} = 0$

$$\therefore \cos \frac{m}{n} 2 i \pi = 0.$$

This (see article 187) will never happen, except when m is odd, and n divisible by 4. In this case, by making $i = \frac{n}{4}$, $\cos \frac{m}{n} 2 i \pi = \cos m \frac{\pi}{2} = 0$; and $\sin \frac{m}{n} 2 i \pi = \sin m \frac{\pi}{2} = \pm 1$; $\pm$ according as m is of the form $4 m' + 1$, or $4 m' + 3$. By making $i = \frac{3n}{4}$, $\cos \frac{m}{n} 2 i \pi = \cos 3 m \frac{\pi}{2} = 0$; and $\sin \frac{m}{n} 2 i \pi = \sin 3 m \frac{\pi}{2} = \mp 1$; $\mp$ according as m is of the form $4 m' + 1$, or $4 m' + 3$.

$$\therefore (2 \cos a)^{\frac{m}{n}} = \pm S_{\frac{n\pi}{2}},$$

$$\text{and } (2 \cos a)^{\frac{m}{n}} = \mp S_{\frac{3n\pi}{2}}.$$

199. Cor. 1. As in art. 188, whatever be the values of m and n, when $i = 0$, $S_{2 i \pi + a} = 0$, and $(2 \cos a)^{\frac{m}{n}} = C_a$. This shows that the common series (191) is confined to arcs less than $\frac{\pi}{2}$ and to the real value of $[2 \cos a]^{\frac{m}{n}}$.

200. Cor. 2. When m is odd and n even, by making $i = \frac{n}{2}$. $\sin \frac{m}{n} 2 i \pi = \sin m \pi = 0$; and $\cos m \pi = -1$, (188)

$$\therefore S_{2 i \pi + a} = 0, \text{ and } (2 \cos a)^{\frac{m}{n}} = -C_{\pi + a}.$$

201. Cor. 3. When m is odd, and $n = 4 n' + 2$.

By making $i = \frac{n-2}{4}$ (see art. 190)

$$\cos\frac{m}{n}(2\,i+1)\,\pi=\cos\frac{m}{2}\,\pi=0\,;\ \text{and}\ \sin\frac{m}{2}\,\pi=\pm 1\,;$$

$\pm$ according as $m = 4\,m' + 1$, or $4\,m' + 3$.

$$C_{(2\,i+1)\,\pi+\alpha}=0,$$

$$(2\cos\alpha)^{\frac{m}{n}}=\pm\ S_{\frac{n}{2}\pi+\alpha}.$$

By making $i = \dfrac{3\,n-2}{4}$

$$\cos\frac{m}{n}(2\,i\,\pi+1)\,\pi=\cos 3\,\frac{m}{2}\,\pi=0;\ \text{and}\sin 3\,\frac{m}{2}\,\pi=\mp 1\,;$$

$\mp$ according as $m = 4\,m' + 1$, or $4\,m' + 3$.

$$\therefore C_{(2i+1)\,\pi+\alpha}=0,$$

$$\text{and}\ (2\cos\alpha)^{\frac{m}{n}}=\mp\ S_{\frac{3n\pi}{2}+\alpha}.$$

202. Cor. 4. When n is odd, (189) by making

$$i=\frac{n-1}{2},\ \sin\frac{m}{n}(2i+1)\,\pi=\sin m\,\pi=0;$$

and $\cos m\,\pi = \pm 1$; $\pm$ according as m is even, or odd;

$$\therefore S_{(2\,i+1)\,\pi+\alpha}=0,$$

$$\text{and}\ (2\cos\alpha)^{\frac{m}{n}}=\pm\ C_{n\,\pi+\alpha}.$$

203. Prop.

$$[x+\sqrt{(x^2-1)}]^{\frac{m}{n}}=[\sqrt{-1}]^{\frac{m}{n}}\,X+\frac{m}{n}\,[\sqrt{-1}]^{\frac{m}{n}-1}\,X'.$$

Where

$$X=1-\frac{m^2}{n^2}\cdot\frac{x^2}{1.\,2}+\frac{m^2}{n^2}\left(\frac{m^2}{n^2}-2^2\right)\cdot\frac{x^4}{1.2.3.4}-\dots$$

and

$$X'=x-\left(\frac{m^2}{n^2}-1^2\right)\cdot\frac{x^3}{1.\,2.\,3}+\left(\frac{m^2}{n^2}-1^2\right)\cdot\left(\frac{m^2}{n^2}-3^2\right)\cdot\frac{x^5}{1.2.3.4.5}-\dots$$

For, let $u = [x + \sqrt{x^2-1}]^{\frac{m}{n}}$

$$=A_0+A_1 x+A_2 x^2+\ldots+A_r x^r+A_{r+1} x^{r+1}+A_{r+1} x^{r+2}+\ldots$$

$$\therefore \frac{du}{dx} = \frac{m}{n} \frac{[x+\sqrt{x^2-1}]^{\frac{m}{n}}}{\sqrt{x^2-1}} = \frac{m}{n} \frac{u}{\sqrt{x^2-1}}$$

$$\therefore (x^2-1) \frac{du^2}{dx^2} - \frac{m^2}{n^2} u^2 = 0$$

$$\therefore (x^2-1) \frac{d^2u}{dx^2} + x \frac{du}{dx} - \frac{m^2}{n^2} u = 0.$$

But,

$$\frac{du}{dx} = \ldots + rA_r x^{r-1} + (r+1)A_{r+1}x^r + (r+2)A_{r+2}x^{r+1} + \ldots$$

$$\frac{d^2u}{dx^2} = \ldots + (r^2-r)A_r x^{r-2} + (r^2+r)A_{r+1}x^{r-1} + (r+1)(r+2)A_{r+2}x^r + \ldots$$

$$\therefore (x^2-1)\frac{d^2u}{dx^2} + x\frac{du}{dx} - \frac{m^2}{n^2}u =$$

$$\ldots + \left((r^2-\frac{m^2}{n^2})A_r - (r+1)(r+2)A_{r+2}\right)x^r + \ldots = 0,$$

$$\therefore A_{r+2} = -A_r . \frac{\left(\frac{m^2}{n^2} - r^2\right)}{(r+1)(r+2)}$$

$$\therefore A_2 = -A_0 . \frac{\frac{m^2}{n^2}}{1.2}$$

$$A_4 = A_0 . \frac{\frac{m^2}{n^2} . \left(\frac{m^2}{n^2} - 2^2\right)}{1.2.3.4}$$

$$A_6 = -A_0 . \frac{\frac{m^2}{n^2} . \left(\frac{m^2}{n^2} - 2^2\right)\left(\frac{m^2}{n^2} - 4^2\right)}{1.2.3.4.5.6}$$

$$\ldots = \ldots\ldots$$

$$A_3 = -A_1 . \frac{\left(\frac{m^2}{n^2} - 1^2\right)}{2.3}$$

$$A_5 = \quad A_1. \frac{\left(\frac{m^2}{n^2} - 1^2\right).\left(\frac{m^2}{n^2} - 3^2\right)}{1.\ 2.\ 3.\ 4.\ 5}$$

$$A_6 = -A_1. \frac{\left(\frac{m^2}{n^2} - 1^2\right).\left(\frac{m^2}{n^2} - 3^2\right).\left(\frac{m^2}{n^2} - 5^2\right)}{1.\ 2.\ 3.\ 4.\ 5.\ 6.\ 7}$$

$$\ldots = \ldots\ldots\ldots\ldots$$

But, by making $x = 0$ in u, and $\frac{du}{dx}$

$$A_0 = [\sqrt{-1}]^{\frac{m}{n}} \text{ and } A_1 = \frac{m}{n}[\sqrt{-1}]^{\frac{m}{n}-1}$$

$$\therefore [x + \sqrt{x^2-1}]^{\frac{m}{n}} = [\sqrt{-1}]^{\frac{m}{n}} X + \frac{m}{n}[\sqrt{-1}]^{\frac{m}{n}-1} X'.$$

204. Cor. If $u = [\sqrt{1-y^2} + y\sqrt{-1}]^{\frac{m}{n}}$

$$(y^2-1)\frac{d^2u}{dy^2} + y\frac{du}{dy} - \frac{m^2}{n^2}u = 0,$$

and by a process similar to that employed in the proposition,

$$[\sqrt{1-y^2} + y\sqrt{-1}]^{\frac{m}{n}} = [1]^{\frac{m}{2n}} Y + \frac{m}{n}[1]^{\frac{m-n}{2n}}\sqrt{-1}\, Y',$$

where $Y = 1 - \frac{m^2}{n^2}\frac{y^2}{1.2} + \frac{m^2}{n^2}.\left(\frac{m^2}{n^2} - 2^2\right).\frac{y^4}{1.2.3.4} - \ldots$

$$Y' = y - \left(\frac{m^2}{n^2} - 1^2\right)\frac{y^3}{1.2.3} + \ldots\ldots$$

205. Prop. *When m is an integer of the form*

$4m'$	$\cos m\alpha = C,$
$4m' + 1,$	$\cos m\alpha = m C',$
$4m' + 2,$	$\cos m\alpha = -C,$
$4m' + 3,$	$\cos m\alpha = -m C'.$

Where $C = 1 - m^2.\frac{(\cos\alpha)^2}{1.\ 2.} + m^2.(m^2-2^2).\frac{(\cos\alpha)^4}{1.2.3.4} - \ldots$

$$C' = \cos a - (m^2 - 1^2)\frac{(\cos a)^3}{1.\,2.\,3} + \dots$$

and a = *any arc whatever.*

For, $\cos m\,a + \sqrt{-1}\sin m\,a = (\cos a + \sqrt{-1}\sin a)^m$
$= (\cos a + \sqrt{(\cos a)^2 - 1})^m$
$= (\sqrt{-1})^m C + m(\sqrt{-1})^{m-1}C'.$

But, if $m = 4\,m'$,
$(\sqrt{-1})^m = (-1)^{2m'} = 1$
$(\sqrt{-1})^{m-1} = -\sqrt{-1}$,

$\therefore$ by equating real quantities,

$$\cos m\,a = C.$$

If $m = 4\,m' + 1$

$$(\sqrt{-1})^m = (\sqrt{-1})^{4m'}\sqrt{-1} = \sqrt{-1}$$

$$(\sqrt{-1})^{m-1} = \frac{(\sqrt{-1})^m}{\sqrt{-1}} = 1$$

$\therefore$ by equating real quantities

$$\cos m a = m\,C'.$$

Similarly for the other cases.

206. Cor. The formulæ for $\sin m\,a$ obtained in the same manner would have been, when m is of the form,

$4\,m'$,	$\sin m\,a = -m\,C'$,
$4\,m' + 1$,	$\sin m\,a = C$,
$4\,m' + 2$,	$\sin m\,a = m\,C'$,
$4\,m' + 3$,	$\sin m\,a = -C$.

These formulæ are useless, since the series C only terminates when m is even, and C' when it is odd. But, by differentiating the equations for the cosine, when m is of the form,

$$4\,m',\ \sin m\,a = -m\sin a\left\{\cos a - \frac{(m^2-2^2)}{1.2.3}(\cos a)^3 + \dots\right\},$$

$$4\,m' + 1,\ \sin m\,a = \sin a\left\{1 - \frac{(m^2-1^2)}{1.\,2}(\cos a)^2 + \dots\right\},$$

$$4\,m' + 2,\ \sin m\,\alpha = m \sin \alpha \left\{ \cos \alpha - \frac{(m^2 - 2^2)}{1.2.3} (\cos \alpha)^3 + \ldots \right\},$$

$$4\,m' + 3,\ \sin m\,\alpha = -\sin \alpha \left\{ 1 - \frac{(m^2 - 1^2)}{1.\ 2} (\cos \alpha)^2 + \ldots \right\},$$

which formulæ have not the same defect, but all of them terminate.

207. PROP.

$$Cos\,\frac{m}{n}\,(2\,i\,\pi \pm \alpha) = C\ cos\frac{m}{n}\,(4\,i \pm 1)\frac{\pi}{2} + \frac{m}{n}\,C' cos\left(\frac{m}{n} - 1\right)(4\,i \pm 1)\frac{\pi}{2}$$

and

$$sin\ \frac{m}{n}\,(2\,i\,\pi \pm \alpha) = C\,sin\,\frac{m}{n}\,(4\,i \pm 1)\frac{\pi}{2} + \frac{m}{n}\,C' sin\left(\frac{m}{n} - 1\right)(4\,i \pm 1)\,\frac{\pi}{2}$$

where $C = 1 - \frac{m^2}{n^2}\,\frac{(cos\ \alpha)^2}{1.\ \ 2} + \frac{m^2}{n^2}.\ \left(\frac{m^2}{n^2} - 2^2\right).\ \frac{(cos\ \alpha)^2}{1.\ 2.\ 3.\ 4} - \ldots$

$$C' = cos\ \alpha - \left(\frac{m^2}{n^2} - 1^2\right)\,\frac{(cos\ \alpha)^3}{1.\ 2.\ 3} + \ldots$$

α = *any positive arc not* $> \frac{\pi}{2}$

and i = *any integer from* 0 *to* $n - 1$.

For, *firstly,*

$$\text{Cos}\,\frac{m}{n}\,(2\,i\,\pi + \alpha) + \sqrt{-1}\ \sin\,\frac{m}{n}(1\,i\,\pi + \alpha) =$$

$$[\cos\alpha + \sqrt{-1}\ \sin\ \alpha]^{\frac{m}{n}} = C[\sqrt{-1}]^{\frac{m}{n}} + \frac{m}{n}\ C'[\sqrt{-1}]^{\frac{m}{n}-1}$$

$$= C\left(\ \cos\ \frac{m}{n}\ (4\,k + 1)\frac{\pi}{2} + \sqrt{-1}\ \sin\ \frac{m}{n}(4k+1)\ \frac{\pi}{2}\right) +$$

$$\frac{m}{n}\,C'\left(\cos\ \left(\frac{m}{n} - 1\right)\ (4\,k + 1)\frac{\pi}{2} + \sqrt{-1}\ \sin\left(\frac{m}{n} - 1\right)(4\,k + 1)\ \frac{\pi}{2}\right)$$

k being any integer from 0 to $n - 1$,

$$\therefore\ \cos\ \frac{m}{n}\,(2\,i\,\pi + \alpha) = C\ \cos\frac{m}{n}\ (4\,k + 1)\,\frac{\pi}{2} +$$

$$\frac{m}{n}\,C' \cos\left(\frac{m}{n} - 1\right)\ (4\,k + 1)\ \frac{\pi}{2}$$

$$\sin \frac{m}{n}(2 i \pi + \alpha) = \text{C} \sin \frac{m}{n}(4k + 1)\frac{\pi}{2} +$$
$$\frac{m}{n} \text{C}' \sin \left(\frac{m}{n} - 1\right)(4k + 1)\frac{\pi}{2}.$$

To determine k, let $\alpha = \frac{\pi}{2}$ $\therefore$ $\cos \alpha = 0$

$$\therefore \text{ C} = 1, \text{C}' = 0,$$

$$\therefore \cos \frac{m}{n}(4i + 1)\frac{\pi}{2} = \cos \frac{m}{n}(4k + 1)\frac{\pi}{2},$$

$$\sin \frac{m}{n}(4i + 1)\frac{\pi}{2} = \sin \frac{m}{n}(4k + 1)\frac{\pi}{2};$$

but i and k lie between 0 and $n - 1$, therefore, as in art. 185. $i = k$

$$\therefore \cos \frac{m}{n}(2 i \pi + \alpha) = \text{C} \cos \frac{m}{n}(4i + 1)\frac{\pi}{2} +$$
$$\frac{m}{n} \text{C}' \cos \left(\frac{m}{n} - 1\right)(4i + 1)\frac{\pi}{2}.$$

$$\sin \frac{m}{n}(2 i \pi + \alpha) = \text{C} \sin \frac{m}{n}(4i + 1)\frac{\pi}{2} +$$
$$\frac{m}{n} \text{C}' \sin \left(\frac{m}{n} - 1\right)(4i + 1)\frac{\pi}{2}.$$

Secondly, $\cos \frac{m}{n}(2 i \pi - \alpha) = \cos \frac{m}{n}\{2(n - i)\pi + \alpha\},$

$$\sin \frac{m}{n}(2 i \pi - \alpha) = - \sin \frac{m}{n}\{2(n - i)\pi + \alpha\}.$$

Therefore by substituting, and remarking, that

$$\frac{m}{n}\{4(n - i)\pi + 1\}\frac{\pi}{2} = 2 m \pi - \frac{m}{n}(4i - 1)\frac{\pi}{2},$$

$$\left(\frac{m}{n} - 1\right)\{4(n - i)\pi + 1\}\frac{\pi}{2} = 2(m - n)\pi - \left(\frac{m}{n} - 1\right)(4i - 1)\frac{\pi}{2}$$

it follows, that

$$\cos \frac{m}{n}(2 i \pi - \alpha) = \text{C} \cos \frac{m}{n}(4i - 1)\frac{\pi}{2} +$$
$$\frac{m}{n} \text{C}' \cos \left(\frac{m}{n} - 1\right)(4i - 1)\frac{\pi}{2}$$

$$\sin\frac{m}{n}(2i\pi - a) = C\sin\frac{m}{n}(4i-1)\frac{\pi}{2} + \frac{m}{n}C'\sin\left(\frac{m}{n}-1\right)(4i-1)\frac{\pi}{2}$$

$$\therefore \cos\frac{m}{n}(2i\pi \pm a) = C\cos\frac{m}{n}(4i\pm 1)\frac{\pi}{2} + \frac{m}{n}C'\cos\left(\frac{m}{n}-1\right)(4i\pm 1)\frac{\pi}{2}$$

$$\text{and } \sin\frac{m}{n}(2i\pi \pm a) = C\sin\frac{m}{n}(4i\pm 1)\frac{\pi}{2} + \frac{m}{n}C'\sin\left(\frac{m}{n}-1\right)(4i\pm 1)\frac{\pi}{2}.$$

208. PROP. *To find what values of i will cause $\cos\frac{m}{n}(4i \pm 1)\frac{\pi}{2}$ to vanish.*

In this case $\frac{m}{n}(4i \pm 1) =$ an odd integer.

But $4i \pm 1 =$ an odd integer,

$\therefore$ m and $\frac{4i \pm 1}{n}$ and $\therefore$ n are odd integers.

But i is not $> n - 1$,

$$\therefore \frac{4i \pm 1}{n} \text{ not } > 4,$$

$$\therefore \frac{4i \pm 1}{n} = 1, \text{ or } i = \frac{n \mp 1}{4}$$

$$\text{or } \frac{4i \pm 1}{n} = 3, \therefore i = \frac{3n \mp 1}{4}.$$

Firstly, If $n = 4n' + 1$

$$i = \frac{n-1}{4} = n'$$

$$i = \frac{n+1}{4} = n' + \tfrac{1}{2}$$

$$i = \frac{3n-1}{4} = 3\,n' + \tfrac{1}{2}$$

$$i = \frac{3n+1}{4} = 3\,n' + 1.$$

But because i is an integer, the second and third values must be rejected. For the two remaining values of i the formulæ become

$$\cos \frac{m}{n}\left(\frac{n-1}{2}\,\pi+\alpha\right) = \text{C} \cos m\,\frac{\pi}{2}+\frac{m}{n}\,\text{C}' \cos(m-n)\,\frac{\pi}{2}$$

$$= \frac{m}{n}\;\text{C}' \sin \frac{m\,\pi}{2} = \pm\;\frac{m}{n}\text{C}'$$

$\pm$ according as $m = 4\,m' + 1$, or $4\,m' + 3$

$$\cos \frac{m}{n}\left(\frac{3\,n+1}{2}\,\pi-\alpha\right) = \pm\;\frac{m}{n}\;\text{C}', \text{ as before.}$$

The results show, that the two cases are essentially the same. This might have been known from the consideration, that

$$\frac{m}{n}\left(\frac{3\,n+1}{2}\,\pi - \alpha\right)+\frac{m}{n}\left(\frac{n-1}{2}\;\pi + \alpha\right) 2\,m\,\pi.$$

And consequently the two arcs have the same cosine.

Secondly, If $n = 4\,n' + 3$

$$i = \frac{n-1}{4} = n' + \tfrac{1}{2}$$

$$i = \frac{n+1}{4} = n' + 1$$

$$i = \frac{3n-1}{4} = 3\,n' + 2$$

$$i = \frac{3\,n+1}{4} = 3n' + \tfrac{5}{2},$$

$\therefore$ the values required are $\dfrac{n+1}{4}$, and $\dfrac{3\,n-1}{4}$, and for them the formulæ become

$$\cos\frac{m}{n}\left(\frac{n+1}{2}\pi - \alpha\right) = \text{C} \cos m\,\frac{\pi}{2} + \frac{m}{n}\,\text{C}' \cos(m-n)\,\frac{\pi}{2}$$

$$= -\frac{m}{n}\,\text{C}' \sin m\,\frac{\pi}{2} = \mp\,\frac{m}{n}\,\text{C}'$$

$\mp$ according as $m = 4m' + 1$, or $4m' + 3$

$$\cos\frac{m}{n}\left(\frac{3n-1}{2}\pi + \alpha\right) = \frac{m}{n}\,\text{C}' \sin 3m\,\frac{\pi}{2} = \mp\,\frac{m}{n}\text{C}'.$$

These two cases, therefore, are really the same.

209. Prob. *To find what values of i will cause $\cos\left(\frac{m}{n} - 1\right)(4i \pm 1)\frac{\pi}{2}$ to vanish.*

In this case $(m-n)\dfrac{4i \pm 1}{n}$ = an odd integer,

$\therefore$ as in the last proposition

$m - n$, and $\dfrac{4i \pm 1}{n}$ and n are odd integers,

and $\therefore$ m an even integer.

Hence when $n = 4n' + 1$, the values of i are

$\dfrac{n-1}{4}$ and $\dfrac{3n+1}{4}$. The results are

$$\cos\frac{m}{n}\left(\frac{n-1}{2}\pi + \alpha\right) = \pm\,\text{C},$$

$\pm$ according as $m = 4m'$, or $4m' + 2$

$$\cos\frac{m}{n}\left(\frac{3n+1}{2}\pi - \alpha\right) = \pm\,\text{C}.$$

Therefore both cases are the same.

When $n = 4n' + 3$, the values of i are

$\dfrac{n+1}{4}$ and $\dfrac{3n-1}{4}$. The results are

$$\cos\frac{m}{n}\left(\frac{n+1}{2} - \alpha\right) = \pm\,\text{C}$$

$$\cos\frac{m}{n}\left(\frac{3n-1}{2}+a\right)=\pm\text{ C}.$$

Both cases are therefore the same.

210. Prob. *To find what values of i will cause* $\sin\frac{m}{n}(4i\pm 1)\frac{\pi}{2}$ *to vanish.*

In this case $m.\frac{(4i\pm 1)}{n}=$ an even integer,

$\therefore$ m is even, and $\therefore$ n is odd, and

$$\frac{4i\pm 1}{n}=\text{an odd integer.}$$

Therefore, as in the former cases, the results are,

when $n=4n'+1$,

$$\sin\frac{m}{n}\left(\frac{n-1}{2}\pi+a\right)=\pm\frac{m}{n}\text{ C}',$$

$$\sin\frac{m}{n}\left(\frac{3n+1}{2}\pi-a\right)=\pm\frac{m}{n}\text{ C}'.$$

The same case.

When $n=4n'+3$,

$$\sin\frac{m}{n}\left(\frac{n+1}{2}\pi-a\right)=\pm\frac{m}{n}\text{ C}',$$

$$\sin\frac{m}{n}\left(\frac{3n-1}{2}\pi+a\right)=\pm\frac{m}{n}\text{ C}'.$$

The same case.

211. Prob. *To find what values of i will cause* $\sin\left(\frac{m}{n}-1\right)\left(4i\pm 1\right)\frac{\pi}{2}$ *to vanish.*

In this case $(m-n)\left(\frac{4i\pm 1}{n}\right)=$ an even integer.

$$\therefore \frac{4i \pm 1}{n} = \text{an integer},$$

$\therefore$ n is odd, and $m - n$ must be even,

$\therefore$ m is odd, and the results are,

When $n = 4n' + 1$,

$$\sin \frac{m}{n}\left(\frac{n-1}{2}\pi + a\right) = \pm C,$$

$$\sin \frac{m}{n}\left(\frac{3n+1}{2}\pi - a\right) = \pm C.$$

The same case.

When $n = 4n' + 3$,

$$\sin \frac{m}{n}\left(\frac{n+1}{2}\pi - a\right) = \pm C,$$

$$\sin \frac{m}{n}\left(\frac{3n-1}{2} + a\right) = \pm C.$$

The same case.

212. Prop. *When m is an even integer, $\cos m a = S$,*

when odd, $\cos m a = \frac{dS'}{da}$

when $S = 1 - m^2 \frac{(\sin a)^2}{1.\ 2} + m^2.\ (m^2 - 2^2).\ \frac{(\sin a)^4}{1.2.3.4} - \ldots$

and $S' = \sin a - (m^2 - 1^2).\ \frac{(\sin a)^3}{1.\ 2.\ 3} + \ldots$

$$\operatorname{Cos} m a + \sqrt{-1}\ \sin m a = (\cos a + \sqrt{-1}\ \sin a)^m$$

$$= \sqrt{1 - (\sin a)^2} + \sqrt{-1}\ \sin a)^m$$

$$= [1]^{\frac{m}{2}} S + m\,[1]^{\frac{m-1}{2}} \sqrt{-1}\, S'.$$

By equating real quantities,

$$\cos m a = [1]^{\frac{m}{2}}\ S = S, \text{ if } m \text{ be even.}$$

If m be odd, this series is useless, since it will never terminate. But, by equating imaginary quantities,

$$\sin m\,\alpha = m\,[1]^{\frac{m-1}{2}}\,S',$$

$$\therefore\ \cos m\,\alpha = [1]^{\frac{m-1}{2}}\,\frac{dS'}{d\alpha} = \frac{dS'}{d\alpha},\ \text{when } m \text{ is odd},$$

$$= \cos\alpha\left(1 - (m^2 - 1^2)\,\frac{(\sin\alpha)^2}{1.\,2} + \cdot\cdot\right),$$

which formula will terminate.

213. Cor. The corresponding formulæ for the sine are, when m is even, $\sin m\,\alpha = -\frac{1}{m}\cdot\frac{dS}{d\alpha}$

$$= m\cos\alpha\,\{\sin\alpha - (m^2 - 2^2).\,\frac{(\sin\alpha)^3}{1.\,2.\,3} + \ldots\},$$

and when m is odd, $\sin m\,\alpha = m\,S'$.

214. Prop.

$$Cos\,\frac{m}{n}\,(2i\,\pi \pm \alpha) = S\,cos\,\frac{m}{n}\,2i\pi \mp \frac{m}{n}\,S'\,sin\left(\frac{m}{n} - 1\right)2i\pi,$$

and

$$sin\,\frac{m}{n}\,(2\,i\,\pi \pm \alpha) = \frac{m}{n}S'cos\left(\frac{m}{n} - 1\right)2i\,\pi \pm S\,sin\,\frac{m}{n}2i\pi,$$

$$where\,S = 1 - \frac{m^2}{n^2}\,\frac{(sin\ \alpha)^2}{1.\,2} + \frac{m^2}{n^2}\left(\frac{m^2}{n^2} - 2^2\right)\frac{(sin\ \alpha)^4}{1.2.3.4} - \ldots$$

$$S' = sin\ \alpha - \left(\frac{m^2}{n^2} - 1^2\right)\frac{(sin\ \alpha)^3}{1.\ 2.\ 3} + \ldots:$$

$$\alpha = any\ positive\ arc\ not > \frac{\pi}{2},$$

and i = *any integer from* 0 *to* $n - 1$.

For, *firstly*,

$$\cos\frac{m}{n}\,(2i\pi + \alpha) + \sqrt{-1}\,\sin\,(2\,i\,\pi + \alpha) = [\sqrt{1 - (\sin\alpha)^2} + \sqrt{-1}\,\sin\alpha]^{\frac{m}{n}}$$

$$= [1]^{\frac{m}{2n}} \text{S} + \frac{m}{n} \sqrt{-1}\, [1]^{\frac{1}{2}\left(\frac{m}{n}-1\right)} \text{S}',$$

$$= \text{S}\left(\cos \frac{m}{n} k\pi + \sqrt{-1} \sin \frac{m}{n} k\pi\right)$$

$$+ \frac{m}{n} \text{S}' \left\{ \sqrt{-1} \cos\left(\frac{m}{n} - 1\right) k\pi - \sin\left(\frac{m}{n} - 1\right) k\pi \right\},$$

k being any integer from 0 to $n - 1$.

$$\therefore \cos \frac{m}{n}(2i\pi + \alpha) = \text{S} \cos \frac{m}{n} k\pi - \frac{m}{n} \text{S}' \sin\left(\frac{m}{n} - 1\right) k\pi,$$

$$\sin \frac{m}{n}(2i\pi + \alpha) = \text{S} \sin \frac{m}{n} k\pi + \frac{m}{n} \text{S}' \cos\left(\frac{m}{n} - 1\right) k\pi.$$

Let $\alpha = 0$, $\therefore \sin \alpha = 0$,

$\therefore \text{S} = 1, \quad \text{S}' = 0,$

$$\therefore \cos \frac{m}{n} 2i\pi = \cos \frac{m}{n} 2k\pi$$

and $\sin \frac{m}{n} 2i\pi = \sin \frac{m}{n} 2k\pi$

$$\therefore 2i = 2k,$$

$$\therefore \cos \frac{m}{n}(2i\pi + \alpha) = \text{S} \cos \frac{m}{n} 2i\pi - \frac{m}{n} \text{S}' \sin\left(\frac{m}{n} - 1\right) 2i\pi,$$

$$\sin \frac{m}{n}(2i\pi + \alpha) = \text{S} \sin \frac{m}{n} 2i\pi + \frac{m}{n} \text{S}' \cos\left(\frac{m}{n} - 1\right) 2i\pi.$$

Secondly,

$$\cos \frac{m}{n}(2i\pi - \alpha) = \cos \frac{m}{n} \{2(n-i)\pi + \alpha\},$$

$$\sin \frac{m}{n}(2i\pi - \alpha) = -\sin \frac{m}{n} \{2(n-i)\pi + \alpha\}.$$

$$\frac{m}{n} 2(n-i)\pi = 2m\pi - \frac{m}{n} 2i\pi,$$

$$\left(\frac{m}{n} - 1\right) 2(n-i)\pi = 2(m-n)\pi - \left(\frac{m}{n} - 1\right) 2i\pi,$$

$$\therefore \cos \frac{m}{n}(2i\pi - \alpha) = \text{S} \cos \frac{m}{n} 2i\pi + \frac{m}{n} \text{S}' \sin\left(\frac{m}{n} - 1\right) 2i\pi,$$

$$\sin\frac{m}{n}(2i\pi-\alpha)=\frac{m}{n}\,S'\cos\left(\frac{m}{n}-1\right)2i\pi-S\sin\frac{m}{n}2i\pi.$$

. 215. Cor. 1. Similarly,

$$\cos\frac{m}{n}\{(2i+1)\pi\pm 1\}=S\cos\frac{m}{n}(2i+1)\pi\mp\frac{m}{n}S'\sin\left(\frac{m}{n}-1\right)(2i+1)\pi$$

$$\sin\frac{m}{n}\{(2i+1)\pi\pm\alpha\}=\frac{m}{n}S'\cos\left(\frac{m}{n}-1\right)(2i+1)\pi\pm S\sin\frac{m}{n}(2i+1)\pi.$$

216. Cor. 2. To find when the coefficients of S and S′ vanish, see articles 187, &c.

217. Prop.

$$2\cos m\alpha=(2\cos\alpha)^m-m(2\cos\alpha)^{m-2}+\frac{m(m-3)}{1.\ 2}(2\cos\alpha)^{m-4}-..$$

when m is a positive integer, and cos m α cannot be developed in a series of descending powers of cos α when m is negative, or fractional.

By De Moivre's formula,

$$\cos m\alpha+\sqrt{-1}\ \sin m\alpha=[\cos\alpha+\sqrt{(\cos\alpha)^2-1}]^m$$

$$\cos m\alpha-\sqrt{-1}\ \sin m\alpha=[\cos\alpha-\sqrt{(\cos\alpha)^2-1}]^m$$

Let $\frac{1}{(\cos\alpha)^2}=x$, and $[1+\sqrt{1-x}]^m+[1-\sqrt{1-x}]^m=u$

Then, by M'Laurin's theorem

$$u=(u)+\left(\frac{du}{dx}\right)\cdot\frac{x}{1}+\left(\frac{d^2u}{dx^2}\right)\cdot\frac{x^2}{1.2}+\cdots$$

$$\therefore\ 2\cos m\alpha=[\cos\alpha]^m+\left(\frac{du}{dx}\right)[\cos\alpha]^{m-2}+\left(\frac{d^2u}{dx^2}\right).[\cos\alpha]^{m-4}+..$$

$$\text{But } u=[1+\sqrt{1-x}]^m+[1-\sqrt{1-x}]^m$$

$$\therefore\ -\frac{du}{dx}=\tfrac{1}{2}\left\{\frac{m}{\sqrt{1-x}}\left([1+\sqrt{1-x}]^{m-1}-[1-\sqrt{1-x}]^{m-1}\right)\right\}$$

$$\frac{d^2u}{dx^2} = \frac{1}{2^2}\left\{\frac{m.(m-1)}{1-x}\left([1+\sqrt{1-x}]^{m-2}+[1+\sqrt{1-x}^{m-2}]\right)\right.$$

$$\left. - \frac{m}{(1-x)^{\frac{3}{2}}}\left([1+\sqrt{1-x}]^{m-1}-[1-\sqrt{1-x}]^{m-1}\right)\right\}$$

$$-\frac{d^3u}{dx^3} = \frac{1}{2^3}\left\{\frac{m.(m-1).(m-2)}{(1-x)^{\frac{3}{2}}}\left([1+\sqrt{1-x}]^{m-3}-[1-\sqrt{1-x}]^{m-3}\right)\right.$$

$$-m.(m-1)\left(\frac{2}{(1-x)^2}+\frac{1}{(1-x)^{\frac{3}{2}}}\right)\left([1+\sqrt{1-x}]^{m-2}+[1-\sqrt{1-x}]^{m-2}\right)$$

$$\left.+\frac{3\,m}{(1-x)^{\frac{5}{2}}}\left([1+\sqrt{1-x}]^{m+1}-[1-\sqrt{1-x}]^{m-1}\right)\right\}$$

$$\ldots . = \ldots\ldots$$

$$\therefore (u) = 2^m + 0^m,$$

$$\left(\frac{du}{dx}\right) = -\frac{1}{2}\left\{m(2^{m-1}-0^{m-1})\right\}$$

$$\left(\frac{d^2u}{dx^2}\right) = \frac{1}{2^2}\left\{m.\ (m-1)\ (2^{m-2}+0^{m-2})-m(2^{m-1}-0^{m-1})\right.$$

$$\left(\frac{d^3u}{dx^3}\right) = -\frac{1}{2^3}\left\{m.\ (m-1)\ (m-2)\ (2^{m-3}-0^{m-3})\right.$$

$$\left.-3\,m.\ (m-1)\ (2^{m-2}+0^{m-2})+3\ (2^{m-1}-0^{m-1})\right\}$$

$$\ldots . = \ldots\ldots$$

Now, if m be a positive fraction, and r, $r+1$ be the integers respectively less and greater than m, the lowest index of 0 in the $(r+1)$th term will be > zero. This term, therefore, and all which precede it, will be finite. The succeeding terms will be either $+\infty$, or $-\infty$. The series, therefore, for a fractional value of m, is useless; the same will evidently obtain when m is negative. But when m is a positive integer, the coefficient will vanish before the index of 0 becomes negative. The series will therefore be finite, and, by reduction, becomes

$$\cos m\,a = 2^{m-1} (\cos a)^m - m\,2^{m-3} (\cos a)^{m-2} + \frac{m.\,(m-3)}{1.\,2} 2^{m-5} (\cos a)^{m-4} - \ldots.$$

$$\therefore 2 \cos m\,a = (2 \cos a)^m - m\,(2 \cos a)^{m-2} + \frac{m.\,(m-3)}{1.\,2} (2 \cos a)^{m-4} - \ldots.$$

Since the same result will necessarily be produced as when *cos m a* is developed according to the ascending powers of cos a, any further consideration of this series will be superfluous.

SECTION V.

ON TRIGONOMETRIC SERIES.

218. **Prob.** *To sum this series*
sin $(\alpha + \beta)$ + *sin* $(\alpha + 2\beta) + \dots + $ *sin* $(\alpha + n\beta)$.

$$\text{Cos}\left(\alpha + n\beta - \frac{\beta}{2}\right) - \cos\left(\alpha + n\beta + \frac{\beta}{2}\right)$$

$$= \cos\left(\alpha + (2n - 1)\frac{\beta}{2}\right) - \cos\left(\alpha + (2n + 1)\frac{\beta}{2}\right)$$

$$= 2\sin(\alpha + n\beta).\ \sin\frac{\beta}{2}.$$

Hence, by giving n the values 1, 2, 3 .... n, the following equations are derived.

$$2\sin(\alpha + \beta).\ \sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + 3\frac{\beta}{2}\right)$$

$$2\sin(\alpha + 2\beta)\ \sin\frac{\beta}{2} = \cos\left(\alpha + 3\frac{\beta}{2}\right) - \cos\left(\alpha + 5\frac{\beta}{2}\right)$$

$$2\sin(\alpha + 3\beta)\ \sin\frac{\beta}{2} = \cos\left(\alpha + 5\frac{\beta}{2}\right) - \cos\left(\alpha + 7\frac{\beta}{2}\right)$$

$$\dots\dots = \dots\dots - \dots\dots$$

$$2\sin(\alpha + n\beta)\sin\frac{\beta}{2} = \cos\left(\alpha + (2n - 1)\frac{\beta}{2}\right) - \cos\left(\alpha + (2n + 1)\frac{\beta}{2}\right)$$

therefore, by adding and putting S_n for the sum of the series to n terms

$$2\,S_n\sin\frac{\beta}{2} = \cos\left(\alpha + \frac{\beta}{2}\right) - \cos\left(\alpha + (2n + 1)\frac{\beta}{2}\right)$$

$$= 2 \sin\left(\alpha + (n+1)\frac{\beta}{2}\right). \sin\frac{n\beta}{2} \text{ (96.)}$$

$$\therefore S_n = \frac{\sin\left(\alpha + (n+1)\frac{\beta}{2}\right). \sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

219. Cor. 1. The value of S_∞ cannot be determined. For it is evident that the terms of the series neither increase, nor decrease, *ad infinitum*. And since they are all finite, and are positive, or negative, according to the magnitude of the arc, the series will alternately converge and diverge, and therefore, its sum to an infinite number of terms cannot be expressed.

220. Cor. 2. If the n^{th} term be of such magnitude,

$$\text{that } \alpha + (2n+1)\frac{\beta}{2} = (2i+1)\frac{\pi}{2}$$

$$\cos\left(\alpha + (2n+1)\frac{\beta}{2}\right) = 0$$

$$\therefore S_n = \frac{\cos\left(\alpha + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}.$$

221. Cor. 3. If $\alpha + (2n+1)\frac{\beta}{2} = 2i\pi$

$$\cos\left(\alpha + (2n+1)\frac{\beta}{2}\right) = 1$$

$$\therefore S_n = \frac{\cos\left(\alpha + \frac{\beta}{2}\right) - 1}{2\sin\frac{\beta}{2}}$$

$$= -\frac{\left(\sin\left(\frac{\alpha}{2} + \frac{\beta}{4}\right)\right)^2}{\sin\frac{\beta}{2}}.$$

222. Cor. 4. If $\alpha + (2n + 1)\frac{\beta}{2} = (2i + 1)\pi$

$$\cos \alpha + (2n + 1)\frac{\beta}{2} = -1$$

$$\therefore S_n = \frac{\cos\left(\alpha + \frac{\beta}{2}\right) + 1}{2 \sin \frac{\beta}{2}}$$

$$= \frac{\left(\cos\left(\frac{\alpha}{2} + \frac{\beta}{4}\right)\right)^2}{\sin \frac{\beta}{2}}.$$

223. Cor. 5. Let $\alpha = 0$

$$\therefore \sin \beta + \sin 2\beta + \ldots + \sin n\beta = \frac{\sin (n + 1)\frac{\beta}{2} \cdot \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}.$$

224. Prob. *To sum the series*
Cos $(\alpha + \beta) +$ *cos* $(\alpha + 2\beta) + \ldots +$ *cos* $(\alpha + n\beta)$.

$$2 \cos(\alpha + n\beta). \sin \frac{\beta}{2} = \sin\left(\alpha + (2n+1)\frac{\beta}{2}\right) - \sin\left(\alpha + (2n-1)\frac{\beta}{2}\right)$$

Hence, by giving n the values $1, 2, 3 \ldots n$, the following equations are derived:

$$2\cos(\alpha + \beta). \cos\frac{\beta}{2} = \sin\left(\alpha + 3\frac{\beta}{2}\right) - \sin\left(\alpha + \frac{\beta}{2}\right)$$

$$2\cos(\alpha + 2\beta). \sin\frac{\beta}{2} = \sin\left(\alpha + 5\frac{\beta}{2}\right) - \sin\left(\alpha + 3\frac{\beta}{2}\right)$$

$$2\cos(\alpha + 3\beta) \sin\frac{\beta}{2} = \sin\left(\alpha + 7\frac{\beta}{2}\right) - \sin\left(\alpha + 5\frac{\beta}{2}\right)$$

$$\ldots\ldots \quad = \ldots\ldots \quad - \ldots\ldots$$

$$2\cos(\alpha + n\beta) \sin\frac{\beta}{2} = \sin\left(\alpha + (2n + 1)\frac{\beta}{2}\right) - \sin \alpha + (2n - 1)\frac{\beta}{2}$$

therefore, by addition,

$$2\,S_n.\ \sin\frac{\beta}{2} = \sin\left((\alpha + (2n+1)\frac{\beta}{2}\right) - \sin\left(\alpha + \frac{\beta}{2}\right)$$

$$= 2\cos\left(\alpha + (n+1)\frac{\beta}{2}\right).\ \sin\frac{n\beta}{2}$$

$$\therefore\ S_n = \frac{\cos\left(\alpha + (n+1)\frac{\beta}{2}\right).\ \sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

The same result may immediately be obtained by putting $\frac{\pi}{2} + \alpha$ for α, in art. 218.

225. Cor. 1. For the reasons given in (art. 219.) it is impossible to express the sum of this series to an infinite number of terms.

226. Cor. 2. When $\alpha + (2n+1)\frac{\beta}{2} = i\,\pi$

$$\sin\left(\alpha + (2n+1)\frac{\beta}{2}\right) = 0$$

$$\therefore\ S_n = -\frac{\sin\left(\alpha + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}.$$

227. Cor. 3. When $\alpha + (2n+1)\frac{\beta}{2} = (4i+1)\frac{\pi}{2}$

$$\sin\left(\alpha + (2n+1)\frac{\beta}{2}\right) = 1$$

$$\therefore\ S_n = \frac{1 - \sin\left(\alpha + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{1 + \cos\left(\frac{\pi}{2} + \alpha + \frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\left(\cos\left(\frac{\pi}{4}+\frac{a}{2}+\frac{\beta}{4}\right)\right)^2}{\sin\frac{\beta}{2}}.$$

228. Cor. 4. When $a + (2n+1)\frac{\beta}{2} = (4i+3)\frac{\pi}{2}$

$$\sin\left(a+(2n+1)\frac{\beta}{2}\right) = -1$$

$$\therefore S_n = \frac{1+\sin\left(a+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{1-\cos\left(\frac{\pi}{2}+a+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\left(\sin\left(\frac{\pi}{4}+\frac{a}{2}+\frac{\beta}{4}\right)\right)^2}{\sin\frac{\beta}{2}}.$$

229. Cor. 5. Let $a = 0$

$$\therefore \cos\beta + \cos 2\beta + \ldots + \cos n\beta = \frac{\cos(n-1)\frac{\beta}{2}\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}.$$

230. Prob. *To sum the series*
$\sin(a+\beta) - \sin(a+2\beta) + \ldots \mp \sin(a+n\beta)$
$\mp$ according as m is even, or odd.

In the formula

$$2\sin(a+n\beta)\cos\frac{\beta}{2} = \sin\left(a+(2n+1)\frac{\beta}{2}\right) + \sin\left(a+(2n-1)\frac{\beta}{2}\right)$$

by giving n the values 1. 2. 3 .... n

$$2 \sin (\alpha + \beta) . \cos \frac{\beta}{2} = \sin \left(\alpha + 3\frac{\beta}{2}\right) \sin \left(\alpha + \frac{\beta}{2}\right)$$

$$- 2 \sin (\alpha + 2\beta) . \cos \frac{\beta}{2} = - \sin \left(\alpha + 5\frac{\beta}{2}\right) - \sin \left(\alpha + 3\frac{\beta}{2}\right)$$

$$2 \sin (\alpha + 3\beta) \cos \frac{\beta}{2} = \sin \left(\alpha + 7\frac{\beta}{2}\right) + \sin \left(\alpha + 5\frac{\beta}{2}\right)$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots$$

$$\mp 2 \sin(\alpha + n\beta) \cos \frac{\beta}{2} = \mp \sin \left(\alpha + (2n+1)\frac{\beta}{2}\right) \mp \sin \alpha + (2n-1)\frac{\beta}{2}$$

therefore, by addition,

$$2 S_n \cos \frac{\beta}{2} = \pm \sin \left(\alpha + (2n+1)\frac{\beta}{2}\right) + \sin \left(\alpha + \frac{\beta}{2}\right)$$

$$\therefore S_n = \frac{\mp \sin \alpha + (2n+1)\frac{\beta}{2} + \sin \left(\alpha + \frac{\beta}{2}\right)}{2 \cos \frac{\beta}{2}}.$$

Firstly, If n be even

$$S_n = - \frac{\cos \left(\alpha + (n+1)\frac{\beta}{2}\right) . \sin \frac{n\beta}{2}}{\cos \frac{\beta}{2}}.$$

Secondly, If n be odd

$$S_n = \frac{\sin \left(\alpha + (n+1)\frac{\beta}{2}\right) \cos \frac{n\beta}{2}}{\cos \frac{\beta}{2}}.$$

231. Cor. Let $\alpha = 0$

$$\therefore \sin \beta - \sin 2\beta + \ldots - \sin 2n\beta = - \frac{\cos (2n+1)\frac{\beta}{2} \sin n\beta}{\cos \frac{\beta}{2}}$$

$$\sin\beta - \sin 2\beta + \ldots + \sin(2n+1)\beta = \frac{\sin(n+1)\beta.\ \cos(2n+1)\frac{\beta}{2}}{\cos\frac{\beta}{2}}.$$

232. PROB. *To sum the series*
cos $(\alpha + \beta)$ — *cos* $(\alpha + 2\beta)$ + . . . $\mp$ *cos* $(\alpha + n\beta)$
$\mp$ *according as n is even, or odd.*

In the formula

$$2\cos(\alpha+n\beta).\cos\frac{\beta}{2} = \cos\left(\alpha + (2n+1)\frac{\beta}{2}\right) + \cos\left(\alpha + (2n-1)\frac{\beta}{2}\right)$$

by giving n the values 1, 2, 3. . . . . n

$$2\cos(\alpha+\beta)\cos\frac{\beta}{2} = \cos\left(\alpha + 3\frac{\beta}{2}\right) + \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$-2\cos(\alpha+2\beta)\cos\frac{\beta}{2} = -\cos\left(\alpha + 5\frac{\beta}{2}\right) - \cos\left(\alpha + 3\frac{\beta}{2}\right)$$

$$2\cos(\alpha+3\beta)\cos\frac{\beta}{2} = \cos\left(\alpha + 7\frac{\beta}{2}\right) + \cos\left(\alpha + 5\frac{\beta}{2}\right)$$

$$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot = \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

$$\mp 2\cos(\alpha+n\beta)\cos\frac{\beta}{2} = \mp\cos\left(\alpha + (2n+1)\frac{\beta}{2}\right) \mp \cos\left(\alpha + (2n-1)\frac{\beta}{2}\right)$$

therefore, by addition,

$$2\,S_n \cos\frac{\beta}{2} = \mp\cos\left(\alpha + (2n+1)\frac{\beta}{2}\right) + \cos\left(\alpha + \frac{\beta}{2}\right)$$

$$\therefore\ S_n = \frac{\mp\cos\left(\alpha + (2n+1)\frac{\beta}{2}\right) + \cos\left(\alpha + \frac{\beta}{2}\right)}{2\cos\frac{\beta}{2}}.$$

Firstly, If n be even

$$S_n = \frac{\sin\left(\alpha + (n+1)\frac{\beta}{2}\right).\ \sin\frac{n\beta}{2}}{\cos\frac{\beta}{2}}.$$

Secondly, If n be odd

$$S_n = \frac{\cos\left(\alpha + (n+1)\frac{\beta}{2}\right)\cos\frac{n\beta}{2}}{\cos\frac{\beta}{2}}.$$

233. Cor. Let $\alpha = 0$

$$\therefore \cos\beta - \cos 2\beta + \ldots - \cos 2n\beta = \frac{\sin(2n+1)\frac{\beta}{2}\sin n\beta}{\cos\frac{\beta}{2}}$$

$$\cos\beta - \cos 2\beta + \ldots + \cos(2n+1)\beta = \frac{\cos(n+1)\beta\cos(2n+1)\frac{\beta}{2}}{\cos\frac{\beta}{2}}.$$

234. Prob. *To sum the series*
$x \sin(\alpha+\beta) + x^2 \sin(\alpha+2\beta) + \ldots + x^n \sin(\alpha + n\beta).$

$\text{Sin}(\alpha + n\beta + \beta) + \sin(\alpha + n\beta + \beta) = 2\sin(\alpha + n\beta)\cos\beta,$

$$\therefore 2x^{n+1}\sin(\alpha+n\beta)\cos\beta = x^{n+1}\sin\{\alpha+(n+1)\beta\} + x^{n+1}\sin\{\alpha+(n-1)\beta\}.$$

Hence, by giving n the values 1, 2, 3 . . . . . n,

$$2x^2\sin(\alpha+\beta)\cos\beta = x^2\sin(\alpha+2\beta) + x^2\sin\alpha,$$
$$2x^3\sin(\alpha+2\beta)\cos\beta = x^3\sin(\alpha+3\beta) + x^3\sin(\alpha+\beta),$$
$$2x^4\sin(\alpha+3\beta)\cos\beta = x^4\sin(\alpha+4\beta) + x^4\sin(\alpha+2\beta),$$
$$\cdots\cdots = \cdots\cdots + \cdots\cdots$$
$$2x^{n+1}\sin(\alpha+n\beta)\cos\beta = x^{n+1}\sin\{\alpha+(n+1)\beta\} + x^{n+1}\sin\{\alpha+(n-1)\beta\}.$$

Therefore, by addition,

$$2x S_n\cos\beta = S_n - x\sin(\alpha+\beta) + x^{n+1}\sin\{\alpha+(n+1)\beta\} + x^2 S_n + x^2\sin\alpha - x^{n+2}\sin\{\alpha+n\beta\}$$

$$\therefore S_n(1-2x\cos\beta+x^2) = x\sin(\alpha+\beta) - x^2\sin\alpha - x^{n+1}\sin\{\alpha+(n+1)\beta\} + x^{n+2}\sin(\alpha+n\beta),$$

$$\therefore S_n = \frac{x \sin(\alpha+\beta) - x^2 \sin\alpha - x^{n+1} \sin\{\alpha+(n+1)\beta\} + x^{n+2} \sin(\alpha+n\beta)}{1 - 2x\cos\beta + x^2}.$$

235. Cor. 1. By differentiating and multiplying by x, may be summed the series

$$x \sin(\alpha+\beta) + 2x^2 (\sin \alpha + 2\beta) + 3x^3 \sin(\alpha+3\beta) + \ldots$$

And, by repeating the operation, the series

$$x \sin(\alpha+\beta) + 2^m x^2 \sin(\alpha+\beta) + 3^m x^3 \sin(\alpha+3\beta) + \ldots$$

may be summed.

236. Cor. 2. By multiplying by x^{r-1}, differentiating and dividing by x^{r-2}, may be summed the series

$$r x \sin(\alpha+\beta) + (r+1) x^2 \sin(\alpha+2\beta) + \ldots.$$

And, by repeating the operation,

$$r^m x \sin(\alpha+\beta) + (r+1)^m x^2 \sin(\alpha+2\beta) + \ldots$$

may be summed.

237. Cor. 3. By dividing by x, and integrating, may be summed

$$x \sin(\alpha+\beta) + \frac{x^2}{2} \sin(\alpha+2\beta) + \frac{x^3}{3} \sin(\alpha+3\beta) + \ldots$$

And, by repeating the operation, the series

$$x \sin(\alpha+\beta) + \frac{x^2}{2^m} \sin(\alpha+2\beta) + \frac{x^3}{3^m} \sin(\alpha+2\beta) + \ldots$$

238. Cor. 4. By dividing by x^{r+2}, integrating and multiplying by x^{r+1}, may be summed the series

$$\frac{x}{r} \sin(\alpha+\beta) + \frac{x^2}{r+1} \sin(\alpha+2\beta) + \ldots.$$

And, by repeating the operation, the series

$$\frac{x}{r^m}\sin(\alpha+\beta)+\frac{x^2}{(r+1)^m}\sin(\alpha+2\beta)+\dots$$

may be summed.

239. Cor. 5. These artifices are evidently not confined to the particular case under consideration, but may be extended to any series of the form

$$x\,A_1 + x^2 A_2 + x^3 A_3 + \dots$$

when A_1, A_2, A_3, &c. are not functions of x.

240. Cor. 6. Let $n = 1$,

$$\therefore \sin(\alpha+\beta)+\sin(\alpha+2\beta)+\dots+\sin(\alpha+n\beta)$$

$$=\frac{\sin(\alpha+\beta)-\sin\alpha-\{\sin(\alpha+(n+1)\beta)-\sin(\alpha+n\beta)\}}{2(1-\cos\beta)}$$

$$=\frac{\cos\left(\left(\alpha+\frac{\beta}{2}\right)\sin\frac{\beta}{2}-\cos\left(\alpha+(2n+1)\frac{\beta}{2}\right)\sin\frac{\beta}{2}\right.}{2\left(\sin\frac{\beta}{2}\right)^2}$$

$$=\frac{\cos\left(\left(\alpha+\frac{\beta}{2}\right)-\cos\left(\alpha+(2n+1)\frac{\beta}{2}\right)\right.}{2\sin\frac{\beta}{2}}$$

$$=\frac{\sin\left(\alpha+(n+1)\frac{\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}},\text{ as in (art. 218.)}$$

241. Cor. 7. Let $n = -1$,

$$\therefore \sin(\alpha+\beta)-\sin(\alpha+2\beta)+\dots\mp\sin(\alpha+n\beta)$$

$$=\frac{\sin\alpha+\sin(\alpha+\beta)+(-1)^n\{\sin(\alpha+(n+1)\beta)+\sin(\alpha+n\beta)\}}{2(1+\cos\beta)}$$

$$= -\frac{\sin\left(\alpha + \frac{\beta}{2}\right)\cos\frac{\beta}{2} \pm \sin\left(\alpha + (2\,n+1)\frac{\beta}{2}\right)\cos\frac{\beta}{2}}{2\left(\cos\frac{\beta}{2}\right)^2}$$

$$= \frac{\mp\sin\left(\alpha + (2\,n+1)\frac{\beta}{2}\right) + \sin\left(\alpha + \frac{\beta}{2}\right)}{2\cos\frac{\beta}{2}}, \text{ as in art. 230.}$$

242. Cor. 8. Let $\alpha = 0$ in the proposition,

$$\therefore\ x\sin\beta + x^2\sin 2\beta + \ldots + x^n\sin n\beta$$

$$= \frac{x\sin\beta - x^{n+1}\sin(n+1)\beta + x^{n+2}\sin n\beta}{1 - 2\,x\cos\beta + x^2}.$$

243. Cor. 9. When x *is less than unity,*

$$x\sin\beta + x^2\sin 2\beta + \ldots\ldots = \frac{x\sin\beta}{1-2\,x\cos\beta+x^2}.$$

244. Cor. 10. By article 237,

$$x\sin\beta + \frac{x^2}{2}\sin 2\beta + \ldots\ldots = \tan^{-1}\left(\frac{x-\cos\beta}{\sin\beta}\right) + \frac{\pi}{2} - \beta.$$

245. Prob. *To sum the series*

$$x.\cos(\alpha+\beta) + x^2\cos(\alpha + 2\beta) + \ldots + x^n\cos(\alpha+n\beta).$$

$$\text{Cos}(\alpha+n\beta+\beta) + \cos(\alpha+n\beta-\beta) = 2\cos(\alpha+n\beta)\cos\beta,$$

$$\therefore\ 2\,x^{n+1}(\cos\alpha + n\beta).\cos\beta = x^{n+1}\cos\{\alpha + (n+1)\beta\}$$
$$+ x^{n+1}\cos\{\alpha + (n-1)\beta\}.$$

And, by giving n the values 1, 2, 3, ... n, the series may be summed as in art. 234.

Or the problem may be solved thus.

In article 234, let $\frac{\pi}{2} + \alpha$ be substituted for α,

$$-\{x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\ldots\ldots+x^n\cos(\alpha+n\beta)\}$$

$$=\frac{x^2\cos\alpha-x\cos(\alpha+\beta)+x^{n+1}\cos\{\alpha+(n+1)\beta\}-x^{n+2}\cos(\alpha+n\beta)}{1-2x\cos\beta+x^2}.$$

246. Cor. 1. Let $x=1$,

$$\therefore S_n=-\frac{\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+n\beta)-\cos\{\alpha+(n+1\beta)\}}{2(1-\cos\beta)}$$

$$=-\frac{\sin\left(\alpha+\frac{\beta}{2}\right)\sin\frac{\beta}{2}-\sin\left(\alpha+(2n+1)\frac{\beta}{2}\right).\ \sin\frac{\beta}{2}}{2\left(\sin\frac{\beta}{2}\right)^2}$$

$$=-\frac{\sin\left(\alpha+\frac{\beta}{2}\right)-\sin\left(\alpha+(2n+1)\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$=\frac{\cos\left(\alpha+(n+1)\frac{\beta}{2}\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}},\text{ as in art. 294.}$$

247. Cor. 2. Let $x=-1$,

$$\therefore S_n=\frac{\cos(\alpha+\beta)+\cos\alpha-(-1)^n\{\cos(\alpha+(n+1)\beta)+\cos(\alpha+n\beta)\}}{2(1+\cos\beta)}$$

$$=\frac{\cos\left(\alpha+\frac{\beta}{2}\right)\cos\frac{\beta}{2}\mp\cos\left(\alpha+(2n+1)\frac{\beta}{2}\right)\cos\frac{\beta}{2}}{2\left(\cos\frac{\beta}{2}\right)^2}$$

$$=\frac{\mp\cos\left(\alpha+(2n+1)\frac{\beta}{2}\right)+\cos\left(\alpha+\frac{\beta}{2}\right)}{\cos\frac{\beta}{2}},\text{ as in art. 232.}$$

248. Cor. 3. Let $\alpha=0$;

$\therefore x\cos\beta+x^2\cos 2\beta+\ldots+x^n\cos n\beta$

$$=\frac{x^{n+2}\cos n\beta - x^{n+1}\cos(n+1)\beta + x\cos\beta - x^2}{1 - 2x\cos\beta + x^2}.$$

249. Cor. 4. When x is less than unity,

$$x\cos\beta + x^2\cos 2\beta + \ldots = \frac{x\cos\beta - x^2}{1-2x\cos\beta + x^2}.$$

250. Prob. *To sum the series*

$$2\sin\frac{a}{2}\left(\sin\frac{a}{2^2}\right)^2 + 2^2\sin\frac{a}{2^2}\cdot\left(\sin\frac{a}{2^3}\right)^2 + \ldots + 2^n\sin\frac{a}{2^n}\cdot\left(\sin\frac{a}{2^{n+1}}\right)^2.$$

$$\text{Sin } 2a = 2\sin a.\cos a = 2\sin a\left(1-2(\sin\tfrac{a}{2})^2\right).$$

$$\therefore 4\sin a.\left(\sin\frac{a}{2}\right)^2 = 2\sin a - \sin 2a,$$

$$4.\ 2^n\sin\frac{a}{2^n}\left(\sin\frac{a}{2^{n+1}}\right)^2 = 2^{n+1}\sin\frac{a}{2^n} - 2^n\sin\frac{a}{2^{n-1}}.$$

Therefore, by giving n the values $1, 2, 3 \ldots . n$,

$$4.\ 2\sin\frac{a}{2}\left(\sin\frac{a}{2^2}\right)^2 = 2^2\sin\frac{a}{2} - 2\sin a,$$

$$4.\ 2^2\sin\frac{a}{2^2}\left(\sin\frac{a}{2^3}\right)^2 = 2^3\sin\frac{a}{2^2} - 2^2\sin\frac{a}{2},$$

$$4.\ 2^3\sin\frac{a}{2^3}\left(\sin\frac{a}{2^4}\right)^2 = 2^4\sin\frac{a}{2^3} - 2^3\sin\frac{a}{2^2},$$

$$\ldots\ldots\ldots = \ldots . - \ldots .$$

$$4.\ 2^n\sin\frac{a}{2^n}\left(\sin\frac{a}{2^{n+1}}\right)^2 = 2^{n+1}\sin\frac{a}{2^n} - 2^n\sin\frac{a}{2^{n-1}}.$$

Therefore, by addition,

$$4\ S_n = 2^{n+1}\sin\frac{a}{2^n} - 2\sin a,$$

$$\therefore S_n = \frac{1}{2}\left(2^n\sin\frac{a}{2^n} - \sin a\right),$$

$$\therefore S_\infty = \frac{1}{2}(a - \sin a).$$

251. PROB. *To sum the series*

$\sin \alpha . \cos \beta + \sin 2\alpha . \cos 3\beta + \ldots\ldots + \sin n\alpha \cos (2n-1)\beta.$

$$2 \sin n\alpha . \cos (2n-1)\beta = \sin \{n\alpha + (2n-1)\beta\} + \sin \{n\alpha - (2n-1)\beta\}.$$

Therefore, by giving n the values $1, 2, 3 \ldots\ldots n$,

$$\begin{aligned} 2 \sin \alpha . \cos \beta &= \sin (\alpha + \beta) + \sin (\alpha - \beta), \\ 2\sin 2\alpha . \cos 3\beta &= \sin (2\alpha + 3\beta) + \sin (2\alpha - 3\beta), \\ 2\sin 3\alpha . \cos 5\beta &= \sin (3\alpha + 5\beta) + \sin (3\alpha - 5\beta). \\ \ldots\ldots &= \ldots\ldots + \ldots\ldots \end{aligned}$$

$$2 \sin n\alpha . \cos (2n-1)\beta = \sin \{n\alpha + (2n-1)\beta\} + \sin \{n\alpha - (2n-1)\beta\}.$$

Therefore, by addition,

$$\begin{aligned} 2 S_n = {} & \sin (\alpha + \beta) + \sin \{(\alpha + \beta) + (\alpha + 2\beta)\} + \ldots \\ & + \sin \{\alpha + \beta + (n-1)(\alpha + 2\beta)\} \\ & + \sin (\alpha - \beta) + \sin \{(\alpha - \beta) + (\alpha - 2\beta)\} + \ldots \\ & + \sin \{(\alpha - \beta) + (n-1)(\alpha - 2\beta)\} \end{aligned}$$

$$= \frac{\sin \left\{(\alpha + \beta) + (n-1)\left(\frac{\alpha}{2} + \beta\right)\right\} . \sin n \left(\frac{\alpha}{2} + \beta\right)}{\sin \left(\frac{\alpha}{2} + \beta\right)}$$

$$+ \frac{\sin \left\{(\alpha - \beta) + (n-1)\left(\frac{\alpha}{2} - \beta\right)\right\} . \sin n \left(\frac{\alpha}{2} - \beta\right)}{\sin \left(\frac{\alpha}{2} - \beta\right)}$$

$$\therefore S_n = \frac{\sin \left\{\frac{\alpha}{2} + n\left(\frac{\alpha}{2} + \beta\right)\right\} \sin n \left(\frac{\alpha}{2} + \beta\right)}{\sin \left(\frac{\alpha}{2} + \beta\right)}$$

$$+ \frac{\sin \left\{\frac{\alpha}{2} + n\left(\frac{\alpha}{2} + \beta\right)\right\} \sin n \left(\frac{\alpha}{2} - \beta\right)}{\sin \left(\frac{\alpha}{2} - \beta\right)}.$$

252. Prob. *To sum the series*

$$\sec a.\sec 2a + \sec 2a.\sec 3a + \ldots + \sec na.\sec(n+1)a.$$

$$\text{Tan}\,(n+1)\,a - \tan n\,a = \frac{\sin a}{\cos na.\cos(n+1)a} \text{ art. } 107.$$

$$= \sin a.\sec n\,a.\sec(n+1)\,a.$$

Therefore, by giving n the values $1, 2, 3 \ldots . n$

$$\sin a \sec a.\sec 2\,a = \tan 2\,a - \tan a,$$
$$\sin a.\sec 2\,a.\sec 3\,a = \tan 3\,a - \tan 2\,a,$$
$$\sin a.\sec 3\,a.\sec 4\,a = \tan 4\,a - \tan 3\,a,$$
$$\ldots\ldots\ldots = \ldots - \ldots.$$
$$\sin a \sec n\,a.\sec(n+1)\,a = \tan(n+1)\,a - \tan n\,a.$$

Therefore, by addition,

$$\sin a.\,S_n = \tan(n+1)\,a - \tan a,$$
$$\therefore\ S_n = \sec a\,\{\tan(n+1)\,a - \tan a).$$

253. Prob. *To sum the series*

$$\sec a.\sec 3\,a - \sec 2a.\sec 4\,a + \ldots \mp \sec n\,a.\sec(n+2)\,a.$$

$$\text{Tan}\,(n+2)\,a - \tan n\,a = \frac{\sin 2\,a}{\cos n\,a.\cos(n+2)\,a} \;(107.)$$

$$= \sin 2\,a.\sec n\,a.\sec(n+2)\,a.$$

Therefore, by giving n the values $1, 2, 3 \ldots n,$

$$\sin 2\,a.\sec a.\sec 3\,a = \tan 3\,a - \tan a,$$
$$-\sin 2\,a.\sec 2\,a.\sec 4\,a = -\tan 4\,a + \tan 2\,a,$$
$$\sin 2\,a.\sec 3\,a.\sec 5\,a = \tan 5\,a - \tan 3\,a,$$
$$-\sin 2\,a.\sec 4\,a.\sec 6\,a = -\tan 6\,a + \tan 4\,a,$$
$$\ldots\ldots\ldots = \ldots\ldots\ldots$$
$$\pm\sin 2a.\sec(n-1)\,a.\sec(n+1)\,a = \pm\tan(n+1)\,a - \tan(n-1)\,a$$
$$\mp\sin 2\,a.\sec n\,a.\sec(n+2)\,a = \mp\tan(n+2)\,a - \tan n\,a.$$

Therefore, by addition,

$$\sin 2\,a\,S_n = \tan 2a - \tan a \mp \{\tan(n+2)\,a - \tan(n+1)\,a\}$$

$$\therefore\ 2\sin a.\cos a.S_n = \frac{\sin a}{\cos a.\ \cos 2a} \mp \frac{\sin a}{\cos(n+1)a.\cos(n+2)a}$$

$$\therefore\ S_n = \frac{1}{2}\sec a\,\{\sec a.\ \text{see}\ 2a \mp \sec(n+1)a.\ \sec(n+2)a\}.$$

254. Prob. *To sum the series*

$$(2\ sec\ 2\ a)^2 + (2^2\ sec\ 2^2\ a)^2 + \ldots . (2^n\ sec\ 2^n\ a)^2.$$

$$\left(\frac{2}{\sin 2a}\right)^2 = \frac{1}{(\sin a)^2\,(\cos a)^2}$$

$$= \frac{(\sin a)^2 + (\cos a)^2}{(\sin a)^2.\ (\cos a)^2}$$

$$\therefore\ (2\ \text{cosec}\ 2\ a)^2 = (\text{cosec}\ a)^2 + (\sec a)^2$$

$$\therefore\ (2^n \sec 2^n a)^2 = (2^{n+1}\ \text{cosec}\ 2^{n+1}\ a)^2 - (2^n\ \text{cosec}\ 2^n\ a)^2.$$

Therefore, by giving n the values 1, 2, 3 . . . n,

$$(2\sec 2\ a)^2 = (2^2\ \text{cosec}\ 2^2\ a)^2 - (2\ \text{cosec}\ 2\ a)^2,$$
$$(2^2 \sec 2^2 a)^2 = (2^3\ \text{cosec}\ 2^3\ a)^2 - (2^2\ \text{cosec}\ 2^2\ a)^2,$$
$$(2^3 \sec 2^3 a)^3 = (2^4\ \text{cosec}\ 2^4\ a)^2 - (2^3\ \text{cosec}\ 2^3\ a)^2,$$
$$\ldots\ldots = \ldots\ldots\ldots - \ldots\ldots\ldots$$
$$(2^n \sec 2^n a)^2 = (2^{n+1}\ \text{cosec}\ 2^{n+1}\ a)^2 - (2^n\ \text{cosec}\ 2^n\ a)^2.$$

Therefore, by addition,

$$S_n = (2^{n+1}\ \text{cosec}\ 2^{n+1}\ a)^2 - (2\cos 2\ a)^2.$$

255. Prob. *To sum the series*

$$cosec\ 2\ a + cosec\ 2^2\ a + \ldots + cosec\ 2^n\ a.$$

Cosec $2\ a = \cot a - \cot 2\ a$, art. (84)

$\therefore$ cosec $2^n\ a = \cot 2^{n-1}\ a - \cot 2^n\ a$.

Therefore, by giving n the values 1, 2, 3 . . . . n,

$$\text{cosec}\ 2\ a = \cot a - \cot 2\ a,$$
$$\text{cosec}\ 2^2\ a = \cot 2\ a - \cot 2^2\ a.$$
$$\text{cosec}\ 2^3\ a = \cot 2^2\ a - \cot 2^3\ a,$$
$$\ldots\ldots = \ldots\ldots - \ldots\ldots$$
$$\text{cosec}\ 2^n\ a = \cot 2^{n+1}\ a - \cot 2^n\ a.$$

Therefore, by addition,

$$S_n = \cot \alpha - \cot 2^n \alpha.$$

256. PROB. *To sum the series*

$$\operatorname{cosec} \alpha . \operatorname{cosec} 2\alpha + \operatorname{cosec} 2\alpha . \operatorname{cosec} 3\alpha + \ldots + \operatorname{cosec} n\alpha . \operatorname{cosec} (n+1)\alpha.$$

$$\operatorname{Cot} n\alpha - \cot (n+1)\alpha = \frac{\sin \alpha}{\sin n\alpha . \sin (n+1)\alpha}$$

$$= \sin \alpha . \operatorname{cosec} n\alpha . \operatorname{cosec} (n+1)\alpha.$$

Therefore, by giving n the values $1, 2, 3 \ldots n$,

$$\sin \alpha . \operatorname{cosec} \alpha . \operatorname{cosec} 2\alpha = \cot \alpha - \cot 2\alpha,$$

$$\sin \alpha . \operatorname{cosec} 2\alpha . \operatorname{cosec} 3\alpha = \cot 2\alpha - \cot 3\alpha,$$

$$\sin \alpha . \operatorname{cosec} 3\alpha . \operatorname{cosec} 4\alpha = \cot 3\alpha - \cot 4\alpha,$$

$$\ldots\ldots\ldots = \ldots - \ldots$$

$$\sin \alpha . \operatorname{cosec} n\alpha . \operatorname{cosec} (n+1)\alpha = \cot n\alpha - \cot (n+1)\alpha.$$

Therefore, by addition,

$$\sin \alpha \, S_n = \cot \alpha - \cot (n+1)\alpha,$$

$$\therefore S_n = \sec \alpha \{\cot \alpha - \cot (n+1)\alpha\}.$$

257. PROB. *To sum the series*

$$\left(\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\right)^2 + \left(\frac{1}{2^2}\operatorname{cosec}\frac{\alpha}{2^2}\right)^2 + \ldots + \left(\frac{1}{2^n}\operatorname{cosec}\frac{\alpha}{2^n}\right)^2$$

$$(\sec \alpha)^2 = (2 \operatorname{cosec} 2\alpha)^2 - (\operatorname{cosec} \alpha)^2$$

$$\therefore \left(\frac{1}{2^n} \sec \frac{\alpha}{2^n}\right)^2 = \left(\frac{1}{2^{n-1}}\operatorname{cosec}\frac{\alpha}{2^{n-1}}\right)^2 - \left(\frac{1}{2^n}\operatorname{cosec}\frac{\alpha}{2^n}\right)^2$$

Therefore, by giving n the values $1, 2, 3 \ldots n$,

$$\left(\frac{1}{2} \sec \frac{\alpha}{2}\right)^2 = (\operatorname{cosec} \alpha)^2 - \left(\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\right)^2$$

$$\left(\frac{1}{2^2} \sec \frac{\alpha}{2^2}\right)^2 = \left(\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\right)^2 - \left(\frac{1}{2^2}\operatorname{cosec}\frac{\alpha}{2^2}\right)^2$$

$$\left(\frac{1}{2^3} \sec \frac{\alpha}{2^3}\right)^2 = \left(\frac{1}{2^2}\operatorname{cosec}\frac{\alpha}{2^2}\right)^2 - \left(\frac{1}{2^3}\operatorname{cosec}\frac{\alpha}{2^3}\right)^2$$

$$\ldots\ldots = \ldots\ldots - \ldots\ldots$$

$$\left(\frac{1}{2^n}\sec\frac{a}{2^n}\right)^2=\left(\frac{1}{2^{n-1}}\operatorname{cosec}\frac{a}{2^{n-1}}\right)^2-\left(\frac{1}{2^n}\operatorname{cosec}\frac{a}{2^n}\right)^2$$

therefore, by addition,

$$S_n = (\operatorname{cosec} a)^2 - \left(\frac{1}{2^n}\operatorname{cosec}\frac{a}{2^n}\right)^2$$

$$S_\infty = (\operatorname{cosec} a)^2 - \frac{1}{a^2}.$$

258. **Prob.** *To sum the series*

$$\frac{1}{2}\, tan\, \frac{a}{2}+\frac{1}{2^2}\, tan\, \frac{a}{2^2} + \ldots\ldots + \frac{1}{2^n}\tan\frac{a}{2^n}.$$

$$\frac{1}{2}\tan a = \frac{1}{2}\cot a - \cot 2a$$

$$\therefore \frac{1}{2^n}\tan\frac{a}{2^n}=\frac{1}{2^n}\cot\frac{a}{2^n}-\frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}}$$

$\therefore$ by giving n the values 1, 2, 3, ... n,

$$\frac{1}{2}\tan\frac{a}{2}=\frac{1}{2}\cot\frac{a}{2}-\cot a.$$

$$\frac{1}{2^2}\tan\frac{a}{2^2}=\frac{1}{2^2}\cot\frac{a}{2^2}-\frac{1}{2}\cot\frac{a}{2}$$

$$\frac{1}{2^3}\tan\frac{a}{2^3}=\frac{1}{2^3}\cot\frac{a}{2^3}-\frac{1}{2^2}\cot\frac{a}{2^2}$$

$$\ldots\ldots = \ldots\ldots - \ldots\ldots$$

$$\frac{1}{2^n}\tan\frac{a}{2^n}=\frac{1}{2^n}\cot\frac{a}{2^n}-\frac{1}{2^{n-1}}\cot\frac{a}{2^{n-1}}$$

$\therefore$ by addition,

$$S_n = \frac{1}{2^n}\cot\frac{a}{2^n} - \cot a.$$

259. **Cor.** If $n = \infty$ $\cot\frac{a}{2^n}=\dfrac{1}{\tan\frac{a}{2^n}} = \dfrac{2^n}{a}$

$$\therefore S_\infty = \frac{1}{a} - \cot a.$$

260. **Prob.** *To sum the series*

$$\left(\frac{1}{2}\ tan\ \frac{a}{2}\right)^2 + \left(\frac{1}{2^2}\ tan\ \frac{a}{2^2}\right)^2 + \ \ldots\ldots\ + \left(\frac{1}{2^n}\ tan\ \frac{a}{2^n}\right)^2.$$

$2 \cot 2\,a = \cot a - \tan a.$ (art. 88.)

$\therefore\ 2^2 (\cot 2\,a)^2 = (\cot a)^2 - 2 + (\tan a)^2$

$\therefore\ (\tan a)^2 = 2^2 (\cot 2\,a)^2 - (\cot a)^2 - 2$

$$\therefore\ \left(\frac{1}{2^n}\ \tan\ \frac{a}{2^n}\right)^2 = \left(\frac{1}{2^{n-1}}\ \cot\ \frac{a}{2^{n-1}}\right)^2 - \left(\frac{1}{2^n}\ \cot\ \frac{a}{2^n}\right)^2 - \frac{1}{2^{2n-1}}$$

therefore, by giving n the values $1, 2, 3 \ldots n$,

$$\left(\frac{1}{2}\ \tan\ \frac{a}{2}\right)^2 = (\cot a)^2 - \left(\frac{1}{2}\ \cot\ \frac{a}{2}\right)^2 - \frac{1}{2}$$

$$\left(\frac{1}{2^2}\ \tan\ \frac{a}{2^2}\right)^2 = \left(\frac{1}{2}\ \cot\ \frac{a}{2}\right)^2 - \left(\frac{1}{2^2}\ \cot\ \frac{a}{2^2}\right)^2 - \frac{1}{2^3}$$

$$\left(\frac{1}{2^3}\ \tan\ \frac{a}{2^3}\right)^2 = \left(\frac{1}{2^2}\ \cot\ \frac{a}{2^2}\right)^2 - \left(\frac{1}{2^3}\ \cot\ \frac{a}{2^3}\right)^2 - \frac{1}{2^5}$$

$$\ldots\ldots = \ldots\ldots - \ldots\ldots - \ldots$$

$$\left(\frac{1}{2^n}\ \tan\ \frac{a}{2^n}\right)^2 = \left(\frac{1}{2^{n-1}}\ \cot\ \frac{a}{2^{n-1}}\right)^2 - \left(\frac{1}{2^n}\ \cot\ \frac{a}{2^n}\right)^2 - \frac{1}{2^{2n-1}}$$

therefore, by addition,

$$S_n = (\cot a)^2 - \left(\frac{1}{2^n}\ \cot\ \frac{a}{2^n}\right)^2 - \frac{2}{3}\left(1 - \frac{1}{2^{2n}}\right).$$

261. **Cor. 1.** $S_\infty = (\cot a)^2 - \dfrac{1}{a^2} - \dfrac{2}{3}.$

262. **Prob.** *To sum the series*

$$\left(\frac{1}{2}\ tan\ \frac{a}{2}\right).\ \left(\frac{1}{2}\ sec\ \frac{a}{2}\right)^2 + \left(\frac{1}{2^2}\ tan\ \frac{a}{2^2}\right)\left(\frac{1}{2^2}\ sec\ \frac{a}{2^2}\right)^2 + \ldots$$

$$+ \left(\frac{1}{2^n}\ tan\ \frac{a}{2^n}\right)\left(\frac{1}{2^n}\ sec\ \frac{a}{2^n}\right)^2.$$

$$\frac{\operatorname{Cos} 2a}{(\sin 2a)^3} = \frac{\{(\cos a)^2 - (\sin a)^2\}\ \{(\cos a)^2 + (\sin a)^2\}}{2^3 (\sin a)^3 (\cos a)^3}$$

$$=\frac{(\cos a)^4-(\sin a)^4}{2^3(\sin a)^3\ (\cos a)^3}=\frac{1}{2^3}\left\{\frac{\cos a}{(\sin a)^3}-\frac{\sin a}{\cos a}.\ \frac{1}{(\cos a)^2}\right\}$$

$$\therefore\ \tan a.\ (\sec a)^2=\frac{\cos a}{(\sin a)^3}-\frac{2^3\cos 2a}{(\sin 2a)^3}$$

$$\therefore\frac{1}{2^n}\ \tan\frac{a}{2^n}\cdot\left(\frac{1}{2^n}\sec\frac{a}{2^n}\right)^2=\frac{\cos\frac{a}{2^n}}{\left(2^n\sin\frac{a}{2^n}\right)^3}-\frac{\cos\frac{a}{2^{n-1}}}{\left(2^{n-1}\cos\frac{a}{2^{n-1}}\right)^3}$$

Therefore, by giving n the values $1, 2, 3 \ldots n$, and by addition

$$S_n=\frac{\cos\frac{a}{2^n}}{\left(2^n\sin\frac{a}{2^n}\right)^3}-\frac{\cos a}{(\sin a)^3}$$

$$S_\infty=\frac{1}{a^3}-\frac{\cos a}{(\sin a)^3}.$$

263. PROB. *To sum the series*

$$2\tan\frac{a}{2}\left(\tan\frac{a}{2^2}\right)^2+2^2\tan\frac{a}{2^2}\left(\tan\frac{a}{2^3}\right)^2+\ldots$$
$$+\ 2^2\tan\frac{a}{2^n}\left(\tan\frac{a}{2^{n-1}}\right)^2.$$

By article 86.

$$\tan\frac{a}{2^n}=\frac{2\tan\frac{a}{2^{n+1}}}{1-\left(\tan\frac{a}{2^{n+1}}\right)^2}$$

$$\therefore\ 2^n\tan\frac{a}{2^n}\cdot\left(\tan\frac{a}{2^{n+1}}\right)^2=2^n\ \tan\frac{a}{2^n}-2^{n+1}\tan\frac{a}{2^{n+1}}$$

Therefore, by giving n the values $1, 2, 3, \ldots n$,

$$2\tan\frac{a}{2}\ \left(\tan\frac{a}{2^2}\right)^2=2\ \tan\frac{a}{2}\ -2^2\tan\frac{a}{2^2}$$

$$2^2\tan\frac{a}{2^2}\ \left(\tan\frac{a}{2^3}\right)^2=2^2\ \tan\frac{a}{2^n}-2^3\ \tan\frac{a}{2^3}$$

$$2^3 \tan\frac{a}{2^3}\left(\tan\frac{a}{2^4}\right)^2 = 2^3 \tan\frac{a}{2^3} - 2^4 \tan\frac{a}{2^4}$$

$$\ldots\ldots \quad = \quad \ldots \quad - \quad \ldots$$

$$2^n \tan\frac{a}{2^n}\left(\tan\frac{a}{2^{n+1}}\right)^2 = 2^n \tan\frac{a}{2^n} - 2^{n+1} \tan\frac{a}{2^{n+1}}$$

Therefore, by addition,

$$S_n = 2\tan\frac{a}{2} - 2^{n+1}\tan\frac{a}{2^{n+1}}$$

$$S_\infty = 2\tan\frac{a}{2} - a.$$

264. Prob. *To sum the series*
$(\tan 2a + \cot 2a) + (\tan 2^2 a + \cot 2^2 a) + \ldots + (\tan 2^n a + \cot 2^n a)$.

$$\tan a - \cot a = -2\cot 2a$$

$$\therefore \tan a + \cot a = 2(\cot a - \cot 2a)$$

$$\therefore \tan 2^n a + \cot 2^n a = 2(\cot 2^n a - \cot 2^{n+1} a)$$

Therefore, by giving n the values, 1, 2, 3 . . . n,

$$\begin{aligned}
\tan 2a + \cot 2a &= 2(\cot 2a - \cot 2^2 a)\\
\tan 2^2 a + \cot 2^2 a &= 2(\cot 2^2 a - \cot 2^3 a)\\
\tan 2^3 a + \cot 2^3 a &= 2(\cot 2^3 a - \cot 2^4 a)\\
\ldots\ldots + \ldots\ldots &= \ldots\ldots - \ldots\\
\tan 2^n a + \cot 2^n a &= 2(\cot 2^n a - \cot 2^{n+1} a)
\end{aligned}$$

Therefore, by addition,

$$S_n = 2(\cot 2a - \cot 2^{n+1} a).$$

265. Prob. *To sum the series*

$$2\cos(a+\beta) + \frac{x^2}{1.\,2}\cos(a+2\beta) + \frac{x^3}{1.\,2.\,3}\cos(a+3\beta) + \ldots$$

$$2\cos(a+\beta) = e^{(a+\beta)\sqrt{-1}} + e^{-(a+\beta)\sqrt{-1}}$$

$$2\cos(a+2\beta) = e^{(a+2\beta)\sqrt{-1}} + e^{-(a+2\beta)\sqrt{-1}}$$

$$2 \cos (\alpha+3\beta) = e^{(\alpha+3\beta)\sqrt{-1}} + e^{-(\alpha+3\beta)\sqrt{-1}}$$

$$\ldots\ldots = \ldots\ldots + \ldots\ldots$$

$$2\,S_\infty = e^{\alpha\sqrt{-1}}\left\{xe^{\beta\sqrt{-1}} + \frac{(xe^{\beta\sqrt{-1}})^2}{1.\ 2} + \frac{(xe^{\beta\sqrt{-1}})^3}{1.\ 2.\ 3} + \ldots\right\}$$

$$+\, e^{-\alpha\sqrt{-1}}\left\{xe^{-\beta\sqrt{-1}} + \frac{(xe^{-\beta\sqrt{-1}})^2}{1.\ 2} + \frac{(xe^{-\beta\sqrt{-1}})^3}{1.\ 2.\ 3} + \ldots\right\}$$

$$= e^{\alpha\sqrt{-1}}\left(e^{xe^{\beta\sqrt{-1}}} - 1\right) + e^{-\alpha\sqrt{-1}}\left(e^{xe^{-\beta\sqrt{-1}}} - 1\right)$$

$$= -\,e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}} + e^{\alpha\sqrt{-1}}\,e^{x(\cos\beta+\sqrt{-1}\sin\beta)} + e^{-\alpha\sqrt{-1}}\,e^{x(\cos\beta-\sqrt{-1}\sin\beta)}$$

$$= -\,2\cos\alpha + e^{x\cos\beta}\left\{e^{(\alpha+x\sin\beta)\sqrt{-1}} + e^{-(\alpha+x\sin\beta)\sqrt{-1}}\right\}$$

$$= -\,2\cos\alpha + 2e^{x\cos\beta}\cos(\alpha + x\sin\beta)$$

$$\therefore\ S_\infty = e^{x\cos\beta}\cos(\alpha + x\sin\beta) - \cos\alpha.$$

266. Prob. *To sum the series*

$$x \sin(\alpha+\beta) + \frac{x^2}{1.\ 2}\sin(\alpha+2\beta) + \frac{x^3}{1.\ 2.\ 3}\sin(\alpha+3\beta) + \ldots$$

The sum of this series may be obtained by a process similar to that employed in the last article; or it may be deduced from it by substituting $\frac{\pi}{2} + \alpha$ for α. The result is

$$S_\infty = e^{x\cos\beta}\sin(\alpha + x\sin\beta) - \sin\alpha.$$

267. Prob. *To sum the series*

$$x\cos(\alpha+\beta) + \frac{x.\ (x-1)}{1.\ 2}\cos(\alpha+2\beta) + \frac{x.(x-1).(x-2)}{1.\ 2.\ 3}\cos(\alpha+3\beta) + \ldots$$

By a process similar to that used in art. 265.

$$2\,S_\infty = e^{\alpha\sqrt{-1}}\left\{xe^{\beta\sqrt{-1}} + \frac{x.\ (x-1)}{1.\ 2}(e^{\beta\sqrt{-1}})^2 + \ldots\right\}$$

$$+ e^{-\alpha\sqrt{-1}}\left\{xe^{-\beta\sqrt{-1}} + \frac{x.(x-1)}{1.\ 2}(e^{-\beta\sqrt{-1}})^2 + \dots\right\}$$

$$= e^{\alpha\sqrt{-1}}\left\{(1+e^{\beta\sqrt{-1}})^x - 1\right\}$$

$$+ e^{-\alpha\sqrt{-1}}\left\{(1+e^{-\beta\sqrt{-1}})^x - 1\right\}$$

$$= -e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}} + e^{\alpha\sqrt{-1}}\left(\frac{e^{\frac{\beta}{2}\sqrt{-1}} + e^{-\frac{\beta}{2}\sqrt{-1}}}{e^{-\frac{\beta}{2}\sqrt{-1}}}\right)^x$$

$$+ e^{-\alpha\sqrt{-1}}\left(\frac{e^{\frac{\beta}{2}\sqrt{-1}} + e^{-\frac{\beta}{2}\sqrt{-1}}}{e^{\frac{\beta}{2}\sqrt{-1}}}\right)^x$$

$$= -e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}} + \left(e^{\frac{\beta}{2}\sqrt{-1}} + e^{-\frac{\beta}{2}\sqrt{-1}}\right)^x \left(e^{(\alpha+\frac{x\beta}{2})\sqrt{-1}} + e^{-(\alpha+\frac{x\beta}{2})\sqrt{-1}}\right)$$

$$= 2\left(2\cos\frac{\beta}{2}\right)^x \cos\left(\alpha + \frac{x\beta}{2}\right) - 2\cos\alpha,$$

$$\therefore S_\infty = \left(2\cos\frac{\beta}{2}\right)^x \cos\left(\alpha + \frac{x\beta}{2}\right) - \cos\alpha.$$

268. Cor. By substituting $\frac{\pi}{2} + \alpha$ for α,

$$x.\sin(\alpha+\beta) + \frac{x.(x-1)}{1.\ 2}\sin(\alpha+2\beta) + \frac{x.(x-1).(x-3)}{1.\ 2.\ 3}\sin(\alpha+3\beta) + \dots$$

$$= \left(2\cos\frac{\beta}{2}\right)^x \sin\left(\alpha + \frac{x\beta}{2}\right) - \sin\alpha.$$

269. Prob. *To sum the series*

$$\tan^{-1}\frac{t_1-t_2}{1+t_1 t_2} + \tan^{-1}\frac{t_2-t_3}{1+t_2 t_3} + \dots + \tan^{-1}\frac{t_n-t_{n+1}}{1+t_n.\, t_{n+1}},$$

where $\tan^{-1} t$ represents the arc to radius unity, of which the tangent is t.

$$\text{Tan}(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha.\tan\beta}$$

$$\therefore \alpha-\beta = \tan^{-1}\frac{\tan\alpha - \tan\beta}{1+\tan\alpha.\ \tan\beta}$$

$$\text{Let } \tan\alpha = t_1 \therefore \alpha = \tan^{-1} t_1$$
$$\tan\beta = t_2 \therefore \beta = \tan^{-1} t_2.$$

Therefore,

$$\tan^{-1}\frac{t_1-t^2}{1+t_1\,t_2} = \tan^{-1} t_1 - \tan^{-1} t_2,$$
$$\tan^{-1}\frac{t_2-t_3}{1+t_2\,t_3} = \tan^{-1} t_2 - \tan^{-1} t_3,$$
$$\tan^{-1}\frac{t^3-t_4}{1+t_3\,t_2} = \tan^{-1} t_3 - \tan^{-1} t_4,$$
$$\cdots\cdots = \cdots\cdots$$
$$\tan^{-1}\frac{t_n-t_{n+1}}{1+t_n.t_{n+1}} = \tan^{-1} t_n - \tan^{-1} t_{n+1},$$
$$\therefore S_n = \tan^{-1} t_1 - \tan^{-1} t_{n+1}.$$

270. Prob. *To sum the series*

$$\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{1}{1+2+2^2}+\ldots+\tan^{-1}\frac{1}{1+n+n^2}.$$

$$\text{Tan}^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1} = \tan^{-1}\frac{\frac{1}{n}-\frac{1}{n+1}}{1+\frac{1}{n(n+1)}}$$
$$= \tan^{-1}\frac{1}{1+n+n^2}$$
$$\therefore \tan^{-1}\frac{1}{1+1+1^2} = \tan^{-1}\frac{1}{1} - \tan^{-1}\frac{1}{2}$$
$$\tan^{-1}\frac{1}{1+2+2^2} = \tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{3}$$
$$\cdots\cdots = \cdots\cdots - \cdots$$
$$\tan^{-1}\frac{1}{1+n+n^2} = \tan^{-1}\frac{1}{n} - \tan^{-1}\frac{1}{n+1}$$
$$\therefore S_n = \tan^{-1} 1 - \tan^{-1}\frac{1}{n+1}$$
$$= \frac{\pi}{4} - \tan^{-1}\frac{1}{n+1}.$$

For more series of this kind, which can be summed in the same manner, see Herschel's "Finite Differences."

271. PROB. *To prove that, if* α *be less than* $\frac{\pi}{4}$,

$$\alpha = \tan \alpha - \frac{1}{3}(\tan \alpha)^3 + \frac{1}{5}(\tan \alpha)^5 - \ldots$$

$$2\sqrt{-1}\left\{\tan \alpha - \frac{1}{3}(\tan \alpha)^3 + \frac{1}{5}(\tan \alpha)^5 - \ldots\right\}$$

$$= 2\left\{(\sqrt{-1}\tan \alpha) + \frac{1}{3}(\sqrt{-1}\tan \alpha)^3 + \frac{1}{5}(\sqrt{-1}\tan \alpha)^5 + \ldots\right\}$$

$$= \log_e \frac{1+\sqrt{-1}\tan \alpha}{1-\sqrt{-1}\tan \alpha}$$

$$= \log_e \frac{\cos \alpha + \sqrt{-1}\sin \alpha}{\cos \alpha - \sqrt{-1}\sin \alpha}$$

$$= \log_e e^{2\alpha\sqrt{-1}}$$

$$= 2\alpha\sqrt{-1}.$$

Therefore, by dividing by $2\sqrt{-1}$,

$$\alpha = \tan \alpha - \frac{1}{3}(\tan \alpha)^3 + \frac{1}{5}(\tan \alpha)^5 - \ldots$$

Another Method.

272. Let $S_\infty = \tan \alpha - \frac{1}{3}(\tan \alpha)^3 + \frac{1}{5}(\tan \alpha)^5 - \ldots$

$$\therefore \frac{dS_\infty}{d\alpha} = (\sec \alpha)^2 \{1 - (\tan \alpha)^2 + (\tan \alpha)^4 - \ldots\}$$

$$= (\sec \alpha)^2 . \frac{1}{1+(\tan \alpha)^2} \quad (\because \tan \alpha < 1)$$

$$= 1,$$

$$\therefore S_\infty = \alpha + C.$$

But when $\alpha = 0$, $S_\infty = 0$,

$$\therefore S_\infty = \alpha.$$

273. PROB. *To sum the series*

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots .$$

$$\operatorname{Cos} a = 1 - \frac{a^2}{1.2} + \frac{a^4}{1.2.3.4} - \ldots . \text{ (art. 165.)}$$

$$= \left(1 - \frac{2^2 a^2}{\pi 2}\right)\left(1 - \frac{2^2 a^2}{3\pi^2}\right)\left(1 - \frac{2^2 a^2}{5^2 \pi^2}\right)\ldots \text{ (art.169.)}$$

$$= 1 - \frac{2^2 a^2}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right\}$$

$$+ \frac{2^2 a^4}{\pi^2}\left\{\frac{1}{1^2.3^2} + \frac{1}{1^2.5^2} + \frac{1}{3^2.5^2} + \ldots\right\}$$

$$+ \ldots\ldots\ldots\ldots$$

Therefore, by equating the coefficients of a^2,

$$\frac{2^2}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots .\right\} = \frac{1}{1.2}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots . = \frac{\pi^2}{8}.$$

274. COR. By equating the coefficients of a^4,

$$\frac{1}{1^2.3^2} + \frac{1}{1^2.5^2} + \frac{1}{3^2.5^2} + \ldots = \frac{\pi^4}{96}.$$

275. PROB. *To sum the series*

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

$$\operatorname{Sin} a = a - \frac{a^3}{1.2.3} + \frac{a^5}{1.2.3.4.5} \quad \ldots .$$

$$= a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\pi^2}\right)\ldots .$$

$$= a - \frac{a^3}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots .\right\}$$

$$+ \frac{a^5}{\pi^4}\left\{\frac{1}{1^2.2^2} + \frac{1}{1^2.3^2} + \frac{1}{2^2.3^2} + \dots\right\}$$

$+ \dots\dots$

Therefore, by equating the coefficients of a^3,

$$\frac{1}{\pi^2}\left\{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right\} = \frac{1}{6}$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \dots\dots = \frac{\pi^2}{6}.$$

276. Cor. By equating the coefficients of a^5,

$$\frac{1}{1^2.2^2} + \frac{1}{1^2.3^2} + \frac{1}{2^2.3^2} + \dots = \frac{\pi^4}{120}.$$

277. Prop. *In a plane triangle,*

$$log_e a = log_e c - \left\{\frac{b}{c} \cos A + \frac{b^2}{2c^2} \cos 2A + \frac{b^3}{3c^3} \cos 3A + ..\right\}$$

For, $a^2 = c^2 + b^2 - 2bc \cos A$. (109.)

Let $2 \cos A = x + \frac{1}{x}$

$$\therefore a^2 = c^2 - bc\left(x + \frac{1}{x}\right) + b^2$$

$$= (c - bx)\left(c - b\frac{1}{x}\right)$$

$$\therefore \frac{a^2}{c^2} = \left(1 - \frac{b}{c} x\right)\left(1 - \frac{b}{c} \cdot \frac{1}{x}\right)$$

$$\therefore 2\log_e \frac{a}{c} = -\left\{\frac{b}{c}x + \frac{b^2}{2c^2} x^2 + \frac{b^3}{3c^3} x^3 + \dots\right\}$$

$$- \left\{\frac{b}{c}\frac{1}{x} + \frac{b^2}{2c^2}\frac{1}{x^2} + \frac{b^3}{3c^3}\frac{1}{x^3} + \dots\right\}$$

$$= -2\left\{\frac{b}{c} \cos A + \frac{b^2}{2c^2} \cos 2A + \frac{b^3}{3c^3} \cos 3A + ..\right\}$$

$$\therefore \log_e a = \log_e c - \left\{\frac{b}{c} \cos A + \frac{b^2}{2c^2} \cos 2A + \frac{b^3}{3c^3} \cos 3A + ..\right\}$$

278. PROP. *In a plane triangle,*

$$\mathrm{B} = \frac{b}{c}\sin \mathrm{A} + \frac{b^2}{2c^2}\sin 2\mathrm{A} + \frac{b^3}{3c^3}\sin 3\mathrm{A} + \dots\dots$$

For, let α, β, γ, be the arcs to radius unity corresponding to A, B, C.

$$\therefore \frac{\sin\gamma}{\sin\beta} = \frac{\sin(\alpha+\beta)}{\sin\beta} = \frac{\sin\alpha\cos\beta}{\sin\beta} + \cos\alpha = \frac{c}{b}$$

$$\therefore \frac{\sin\beta}{\cos\beta} = \frac{\frac{b}{c}\sin\alpha}{1 - \frac{b}{c}\cos\alpha}$$

$$\therefore \frac{e^{\beta\sqrt{-1}} - e^{-\beta\sqrt{-1}}}{e^{\beta\sqrt{-1}} + e^{-\beta\sqrt{-1}}} = \frac{\frac{b}{c}\left(e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\right)}{2 - \frac{b}{c}\left(e^{\alpha\sqrt{-1}} + e^{-\alpha\sqrt{-1}}\right)}$$

$$\therefore e^{2\beta\sqrt{-1}} = \frac{1 - \frac{b}{c}e^{-\alpha\sqrt{-1}}}{1 - \frac{b}{c}e^{\alpha\sqrt{-1}}}$$

$$\therefore 2\beta\sqrt{-1} = \log_e\left(1 - \frac{b}{c}e^{-\alpha\sqrt{-1}}\right) - \log_e\left(1 - \frac{b}{c}e^{\alpha\sqrt{-1}}\right)$$

$$= \frac{b}{c}e^{\alpha\sqrt{-1}} + \frac{b^2}{2c^2}e^{2\alpha\sqrt{-1}} + \frac{b^3}{3c^3}e^{3\alpha\sqrt{-1}} + \dots\dots$$

$$- \left\{\frac{b}{c}e^{-\alpha\sqrt{-1}} + \frac{b^2}{2c^2}e^{-2\alpha\sqrt{-1}} + \frac{b^3}{3c^3}e^{-3\alpha\sqrt{-1}} + \dots\dots\right\}$$

$$= 2\sqrt{-1}\left\{\frac{b}{c}\sin\alpha + \frac{b^2}{2c^2}\sin 2\alpha + \frac{b^3}{3c^3}\sin 3\alpha + \dots\right\}$$

$$\therefore \mathrm{B} = \frac{b}{c}\sin \mathrm{A} + \frac{b^2}{2c^2}\sin 2\mathrm{A} + \frac{b^3}{3c^3}\sin 3\mathrm{A} + \dots\dots$$

279. COR. 1. Similarly, if $\tan\alpha = \dfrac{m\tan\beta}{1+m\tan\beta}$

$$\alpha = m \sin \beta - \frac{m^2}{2} \sin 2\beta + \frac{m^3}{3} \sin 3\beta + \ldots$$

280. Cor. 2. If $\tan \alpha = m \tan \beta$,

$$\alpha = \beta + \frac{m-1}{m+1} \sin 2\beta + \frac{1}{2}\left(\frac{m-1}{m+1}\right)^2 \sin 4\beta + \ldots .$$

281. Prob. *To expand* $(a^2 - 2ab \cos \alpha + b^2)^{-m}$ *into a series of the form*

$$A_0 + A_1 \cos \alpha + A_2 \cos 2\alpha + \ldots + A_r \cos r\alpha + \ldots$$

Let $2 \cos \alpha = x + \frac{1}{x}$

$$\therefore (a^2 - 2ab \cos \alpha + b^2) = a^2 - ab\left(x + \frac{1}{x}\right) + b^2$$

$$= (a - bx)\left(a - b\frac{1}{x}\right)$$

$$\therefore (a^2 - 2ab \cos \alpha + b^2)^{-m} = (a - bx)^{-m}\left(a - b\frac{1}{x}\right)^{-m}.$$

Let $\frac{1}{a^m} = a_0, \frac{mb}{a^{m+1}} = a_1, \frac{m.(m+1)}{1.\ 2} \frac{b^2}{a^{m+2}} = a_2$, &c. = &c.

and $\frac{m.(m+1) \ldots . (m+r-1)}{1.\ 2.\ 3.\ \ldots r} . \frac{b^r}{a^{m+r}} = a_r$

$$\therefore (a^2 - 2ab \cos \alpha + b^2)^{-m} = (a_0 + a_1 x + a_2 x^2 + \ldots)$$
$$\times \left(a_0 + a_1 \frac{1}{x} + a_2 \frac{1}{x^2} + \ldots\right),$$

$$= (a_0^2 + a_1^2 + a_2^2 + \ldots)$$
$$+ (a_0 a_1 + a_1 a_2 + a_2 a_3 + \ldots)\left(x + \frac{1}{x}\right)$$
$$+ (a_0 a_2 + a_1 a_3 + a_2 a_4 + \ldots)\left(x^2 + \frac{1}{x^2}\right)$$
$$+ \ldots \ldots$$
$$+ (a_0 a_r + a_1 a_{r+1} + a_2 a_{r+2} + \ldots)\left(x^r + \frac{1}{x^r}\right)$$
$$+ \ldots \ldots$$

But $x + \frac{1}{x} = 2\cos\alpha,\ x^2 + \frac{1}{x^2} = 2\cos 2\alpha,\ x^r + \frac{1}{x^r} = 2\cos r\alpha,$

$$\begin{aligned}
\therefore\ A_0 &= a_0^2 + a_1^2 + a_2^2 + \dots \\
A_1 &= 2(a_0 a_1 + a_1 a_2 + a_2 a_3 + \dots) \\
\dots &= \dots\dots\dots \\
A_r &= 2(a_0 a_r + a_1 a_{r+1} + a_2 a_{r+2} + \dots)
\end{aligned}$$

282. PROB. *To expand* $\frac{1}{(1-p\cos\alpha)^m}$ *into a series of the form*

$$A_0 + A_1\cos\alpha + A_2\cos 2\alpha + \dots$$

$$\text{Let } 2\cos\alpha = x + \frac{1}{x}$$

$$\text{and } 1 - p\cos\alpha = (a - bx)\left(a - b\frac{1}{x}\right)$$

$$\begin{aligned}
\therefore\ a^2 + b^2 &= 1 \\
2ab &= p \\
\therefore\ a + b &= \sqrt{(1+p)} \\
a - b &= \sqrt{(1-p)} \\
\therefore\ 2a &= \sqrt{(1+p)} + \sqrt{(1-p)} \\
2b &= \sqrt{(1+p)} - \sqrt{(1-p)},
\end{aligned}$$

$$\text{and } \frac{1}{(1-p\cos\alpha)^m} = (a - bx)^{-m}\left(a - b\frac{1}{x}\right)^{-m}$$

and the operation is the same as in the last proposition.

283. COR. $\frac{b}{a} = \frac{\sqrt{(1+p)} - \sqrt{(1-p)}}{\sqrt{(1+p)} + \sqrt{(1-p)}}$

$$\therefore\ = \frac{1}{p}\sqrt{\{1 - \sqrt{(1-p^2)}\}}$$

If, therefore, $m = 1$ the equation becomes

$$\frac{1}{1 - p\cos\alpha} = \frac{1}{a^2}\left(\frac{1}{1-cx}\right)\left(\frac{1}{1 - c\frac{1}{x}}\right)$$

$\left(\text{by putting } \dfrac{b}{a} = c\right)$

$$= \frac{1}{a^2}\left\{1 + cx + c^2 x^2 + \dots\right\}$$

$$\times \left\{1 + c\frac{1}{x} + c^2 \frac{1}{x^2} + \dots\right\}$$

$$= \frac{1}{a^2}\left\{(1 + c^2 + c^3 + \dots)\right.$$

$$+ \quad (c + c^3 + c^5 + \dots)\left(x + \frac{1}{x}\right)$$

$$+ \ (c^2 + c^4 + c^6 + \dots)\left(x^2 + 1\frac{1}{x^2}\right)$$

$$\left. + \ . \ . \ . \ . \ . \ . \ . \ . \ . \right\}$$

$$= \frac{1}{a^2 - b^2}\left\{1 + 2c\cos\alpha + 2c^2\cos 2\alpha + \dots\right\}$$

When $a + b = \sqrt{(1 + p)}$

$a - b = \sqrt{(1 - p)}$

$\therefore\ a^2 - b^2 = \sqrt{(1 - p^2)}$.

284. **Prob.** *To expand $log_e = (1 - p\cos\alpha)$ into a series of the form*

$$A_0 + A_1 \cos\alpha + A_2 \cos 2\alpha + \dots$$

Let $2\cos\alpha = x + \dfrac{1}{x}$, and

$$1 - m\cos\alpha = a^2 (1 - cx)\left(1 - c\frac{1}{x}\right)$$

$$\therefore\ a = \frac{1}{2}\left\{\sqrt{(1 + p)} + \sqrt{(1 - p)}\right\}$$

$$c = \frac{1}{p}\left\{1 - \sqrt{(1 - p^2)}\right\}.$$

$$\text{Log}_e (1 - p\cos\alpha) = 2\log_e a + \log_e (1 - cx)$$

$$+ \log_e\left(1 - c\frac{1}{x}\right)$$

$$= 2\log_e a - \left\{ cx + \frac{c^2}{x}x^2 + \frac{c^3}{3}n^3 + \ldots \right\}$$

$$- \left\{ c\,\frac{1}{x} + \frac{c^2}{2}\cdot\frac{1}{x^2} + \frac{c^3}{3}\cdot\frac{1}{x^3} + \ldots \right\}$$

$$= 2\log_e a - 2\left\{ c\cos a + \frac{c^2}{2}\cos 2a + \frac{c^3}{3}\cos 3a + \ldots \right\}.$$

285. PROB. *To resolve $e^x \pm e^{-x}$ into quadratic factors.*

$$e^x = 1 + x + \frac{x^2}{1.\ 2} + \frac{x^3}{1.\ 2.\ 3} + \ldots$$

and $e^{-x} = 1 - x + \dfrac{x^2}{1.\ 2} - \dfrac{x^3}{1.\ 2.\ 3} + \ldots$

$$\therefore \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{1.\ 2} + \frac{x^4}{1.\ 2.\ 3.\ 4} + \ldots$$

Now, by articles 165 and 169,

$$1 - \frac{x^2}{1.\ 2} + \frac{x^4}{1.\ 2.\ 3.\ 4} - \ldots = \cos x$$

$$= \left(1 - \frac{2^2x^2}{\pi^2}\right)\left(1 - \frac{2^2x^2}{3^2\pi^2}\right)\left(1 - \frac{2^2x^2}{5^2\pi^2}\right)\ldots$$

But if this expression, and

$$\left(1 + \frac{2^2x^2}{\pi^2}\right)\left(1 + \frac{2^2x^2}{3^2\pi^2}\right)\left(1 + \frac{2^2x^2}{5^2\pi^2}\right)\ldots$$

be multiplied out, the results will be the same, excepting that in the former case the terms will be alternately positive and negative, and in the latter all the terms will be positive,

$$\therefore 2\left\{1 + \frac{x^2}{1.\ 2} + \frac{x^4}{1.\ 2.\ 3.\ 4} + \ldots\right\} =$$

$$e^x + e^{-x} = 2\left(1 + \frac{2^2x^2}{\pi^2}\right)\left(1 + \frac{2^2x^2}{3^2\pi^2}\right)\ldots$$

Similarly,

$$e^x - e^{-x} = 2x\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\ldots$$

Another Method.

286.

$$e^{a\sqrt{-1}} + e^{-a\sqrt{-1}} = 2\cos a = 2\left(1 - \frac{2^2 a^2}{\pi^2}\right)\left(1 - \frac{2^2 a^2}{3^2 \pi^2}\right)\dots\dots$$

$$e^{a\sqrt{-1}} - e^{-a\sqrt{-1}} = 2\sqrt{-1}\sin a = 2\sqrt{-1}\ a\left(1 - \frac{a^2}{\pi^2}\right)\left(1 - \frac{a^2}{2^2\pi^2}\right)\dots$$

Let $a\sqrt{-1} = x$

$$\therefore\ e^x + e^{-x} = 2\left(1 + \frac{2^2 x^2}{\pi^2}\right)\left(1 + \frac{2^2 x^2}{\pi^2}\right)\dots$$

$$\text{and } e^x - e^{-x} = 2a\left(1 + \frac{x^2}{\pi^2}\right)\left(1 + \frac{x^2}{2^2\pi^2}\right)\dots\dots$$

287. Cor. Since $x = \dfrac{x+y}{2} + \dfrac{x-y}{2}$

and $y = \dfrac{x+y}{2} - \dfrac{x-y}{2}$.

Therefore,

$$e^x \pm e^y = e^{\frac{x+y}{2}}\left(e^{\frac{x-y}{2}} \pm e^{-\frac{x-y}{2}}\right)$$

And x and y may be taken either positive, or negative,

$\therefore\ e^{\pm x} \pm e^{\pm y}$ may be expanded in a similar manner.

288. Prop. *Sin* $m\,a = 2^{m-1}$ *sin* a. *sin* $\left(a + \dfrac{\pi}{m}\right)$. *sin* $\left(a + \dfrac{2\pi}{m}\right)$ .... *to m factors,*

where m is a positive integer, and $m\,a < \pi$.

For, by substituting $2ma$ for a in art. 183, and by making $x = 1$

$$2 - 2\cos 2ma = (2 - 2\cos 2a)\left(2 - 2\cos 2\left(a + \frac{\pi}{m}\right)\right)\dots$$

$$\therefore\ (2\sin m\,a)^2 = 2^{2m}\,(\sin a)^2 \left(\sin\left(a + \frac{\pi}{m}\right)\right)^2 \ldots$$

$$\therefore\ 2\sin m\,a = \pm\, 2^m \sin a.\ \sin\left(a + \frac{\pi}{m}\right) \ldots .$$

But, if $m\,a$ be less than π, the greatest arc in the series, $a + \frac{(m-1)\,\pi}{m} = \pi - \frac{\pi - m\,a}{m}$ is less than π. Therefore, all the factors are positive,

$$\therefore \sin m\,a = 2^{m-1} \sin a.\sin\left(a + \frac{\pi}{m}\right).\ \sin\left(a + \frac{2\pi}{m}\right) \ldots .$$

to m terms.

289. Cor 1. If ma be greater than π, the sign will depend upon the magnitude of ma, and the number of negative factors in the series.

290. Cor. 2.

$$\frac{\operatorname{Sin} m\,a}{\sin a} = 2^{m-1} \sin\left(a + \frac{\pi}{m}\right).\ \sin\left(a + \frac{2\pi}{m}\right) \ldots$$

to $m-1$ factors.

Let $a = 0$

$$\therefore\ m = 2^{m-1} \sin\frac{\pi}{m}.\ \sin\frac{2\pi}{m}.\ \sin\frac{3\pi}{m} \ldots .$$

to $m-1$ factors.

291. Cor. 3. By taking the logarithm, and differentiating

$$m \cot m\,a = \cot a + \cot\left(a + \frac{\pi}{m}\right) + \ldots .$$

to m terms.

292. Cor. 4. By differentiating again,

$$m^2\,(\operatorname{cosec} m\,a)^2 = (\operatorname{cosec} a)^2 + \left(\operatorname{cosec}\left(a + \frac{\pi}{m}\right)\right)^2 + \ldots$$

$$= 1 + (\cot \alpha)^2 + 1 + \left(\cot\left(\alpha + \frac{\pi}{m}\right)\right)^2 + \ldots\ldots$$

$$\therefore\ m^2 (\operatorname{cosec} m\alpha)^2 - m = (\cot \alpha)^2 + \left(\cot\left(\alpha + \frac{\pi}{m}\right)\right)^2 + \ldots$$

to m terms.

293. **Prop.**

$$\left.\begin{array}{l} m \text{ even, } (-1)^{\frac{m}{2}} \sin m\,\alpha \\ m \text{ odd, } (-1)^{\frac{m-1}{2}} \cos m\,\alpha \end{array}\right\} = \left\{\begin{array}{l} 2^{m-1} \cos \alpha . \cos\left(\alpha + \frac{\pi}{m}\right). \cos\left(\alpha + \frac{2\pi}{m}\right)\ldots \\ \text{to } m \text{ factors,} \end{array}\right\}$$

where m is a positive integer and $m\,\alpha < \frac{\pi}{2}$.

For, by substituting $2\,m\,\alpha$ for α, in article 183, and by making $x = -1$,

$$2 \mp 2 \cos 2m\,\alpha =$$

$$(2 + 2 \cos 2\,\alpha).\left(2 + 2 \cos 2\left(\alpha + \frac{\pi}{m}\right)\right) \ldots\ldots$$

$$= 2^{2m} (\cos \alpha)^2. \left(\cos\left(\alpha + \frac{\pi}{m}\right)\right)^2 \ldots\ldots$$

$$\therefore\ \left.\begin{array}{l} m \text{ even, } \pm \sin m\,\alpha \\ m \text{ odd, } \pm \cos m\,\alpha \end{array}\right\} = 2^{m-1} \cos \alpha.\ \cos\left(\alpha + \frac{\pi}{m}\right) \ldots\ldots$$

Now, all the arcs of the series lie between 0 and π, for the greatest $\alpha + \frac{(m-1)\,\pi}{m} = \pi - \frac{\pi - m\,\alpha}{m}$ is less than π. Therefore, all the arcs which are greater than $\frac{\pi}{2}$ will have negative cosines, and the series will be $\pm$ according as the number of arcs greater than $\frac{\pi}{2}$ is even, or odd.

Let l = the number of arcs less than $\frac{\pi}{2}$,

$\therefore\ m-l =$ the number of arcs greater than $\frac{\pi}{2}$.

$\therefore\ \alpha + \frac{(l-1)\pi}{m}$ is the last arc in the series which is less than $\frac{\pi}{2}$,

$\therefore\ 2(l-1)$ is the integer next inferior to $m - \frac{2m\alpha}{\pi}$, $2m\alpha$ being less than π.

Therefore, when m is *even*,

$$2(l-1) = m-2,$$

$$\therefore\ l = \frac{m}{2}$$

$$\therefore\ m-l = \frac{m}{2},$$

$$\therefore\ (-1)^{\frac{m}{2}} \sin m\alpha = 2^{m-1} \cos \alpha . \cos\left(\alpha + \frac{\pi}{m}\right) \ldots .$$

When m is *odd*,

$$2(l-1) = m-1,$$

$$\therefore\ l = \frac{m+1}{2}$$

$$\therefore\ m-l = \frac{m-1}{2},$$

$$\therefore\ (-1)^{\frac{m-1}{2}} \cos m\alpha = 2^{m-1} \cos \alpha . \cos\left(\alpha + \frac{\pi}{m}\right) \ldots\ldots$$

to m factors.

294. Cor. 1. If $m\alpha$ be greater than $\frac{\pi}{2}$, the sign will depend upon the magnitude of $m\alpha$, and the number of negative factors in the series.

295. Cor. 2. Let $\alpha = 0$, therefore, when m is odd,

$$(-1)^{\frac{m-1}{2}} = 2^{m-1} \cos \frac{\pi}{m} . \cos \frac{2\pi}{m} \ldots\ldots$$

to $m - 1$ factors.

296. Cor. 3. $m = 2^{m-1} \sin \frac{\pi}{m} \sin \frac{2\pi}{m} \ldots.$

to $m - 1$ factors.

Therefore, when m is odd, by division,

$$(-1)^{\frac{m-1}{2}} m = \tan \frac{\pi}{m} . \tan \frac{2\pi}{m} \ldots\ldots$$

to $m - 1$ factors.

297. Cor. 4. By taking the logarithm, and differentiating,

When m is even,

$$- m \cot m\, \alpha = \tan \alpha + \tan \left(\alpha + \frac{\pi}{m}\right) + \ldots.$$

to m terms.

When m is odd,

$$m \tan m\, \alpha = \tan \alpha + \tan \left(\alpha + \frac{\pi}{m}\right) + \ldots.$$

to m terms.

298. Cor. 5. By differentiating again,

When m is even,

$$m^2 (\operatorname{cosec} m\, \alpha)^2 = (\sec \alpha)^2 + \left(\sec \left(\alpha + \frac{\pi}{m}\right)\right)^2 + \ldots.$$

$$= 1 + (\tan \alpha)^2 + 1 + \left(\tan \left(\alpha + \frac{\pi}{m}\right)\right)^2 + \ldots.$$

$$\therefore\ m^2 (\operatorname{cosec} m\, \alpha)^2 - m = (\tan \alpha)^2 + \left(\tan \left(\alpha + \frac{\pi}{m}\right)\right)^2 + \ldots.$$

to m terms.

When m is odd,

$$m^2 (\sec a)^2 - m = (\tan a)^2 + \left(\tan (a + \frac{\pi}{m})\right)^2 + \ldots.$$

to m terms.

299. Cor. 6. By articles 291. and 297.

$$m \cot m\, a = \cot a + \cot \left(a + \frac{\pi}{m}\right) + \ldots.$$

$$\left.\begin{array}{l} m \text{ even}, \; -m \cot m\, a = \tan a + \tan \left(a + \frac{\pi}{m}\right) + \ldots. \\ m \text{ odd}, \quad m \tan m\, a = \tan a + \tan \left(a + \frac{\pi}{m}\right) + \ldots. \end{array}\right\}$$

Therefore, by addition,

$$\left.\begin{array}{l} m \text{ even}, \qquad 0 = \operatorname{cosec} 2a + \operatorname{cosec} 2\left(a + \frac{\pi}{m}\right) + \ldots. \\ m \text{ odd}, \; m \operatorname{cosec} 2\, m\, a = \operatorname{cosec} 2a + \operatorname{cosec} 2\left(a + \frac{\pi}{m}\right) + \ldots. \end{array}\right\}$$

to m terms.

By subtraction,

$$\left.\begin{array}{l} m \text{ even}, \quad m \cot m\, a = \cot 2\, a + \cot 2 \left(a + \frac{\pi}{m}\right) + \ldots \\ m \text{ odd}, \quad m \cot 2\, m\, a = \cot 2\, a + \cot 2 \left(a + \frac{\pi}{m}\right) + \ldots \end{array}\right\}$$

to m terms.

SECTION VI.

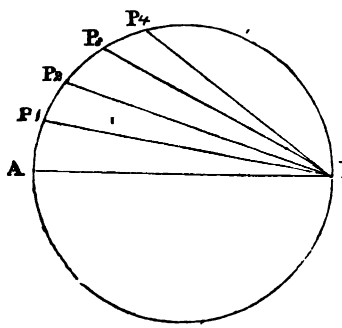
MISCELLANEOUS PROBLEMS.

VIETAS' PROPERTY OF CHORDS.

300. PROP. *If in a circle described with radius unity equal arcs* $AP_1\ P_1P_2,\ P_2P_3\dots$ *be taken, and from the point* B *the opposite extremity of the diameter* AB *the lines* BA, BP_1, BP_2, BP_3, ... *be drawn. Then, in the quadradic equation,*

$$x^2 - px + 1 = 0,$$

in which p is not greater than 2, *if the sum of the roots be equal to the chord* BP_1, *the sums of the squares, cubes, &c. will be equal respectively to the chords* BP_2, BP_3, . . .

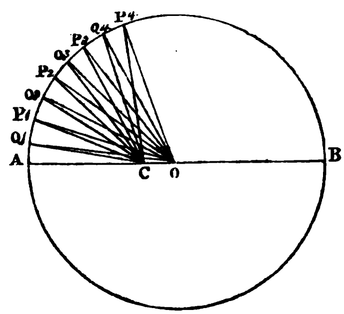


For if x be one root, then by the nature of equations $\frac{1}{x}$ will be the other root.

$$\text{But chord (arc BP}_1) = \text{chord } (\pi - AP_1)$$
$$= 2\sin\left(\frac{\pi}{2} - \frac{AP_1}{2}\right)$$
$$= 2\cos\frac{AP_1}{2}$$
$$\therefore BP_1 = 2 \cos \frac{AP_1}{2} = x + \frac{1}{x}$$
$$\therefore BP_2 = 2\cos 2\frac{AP_1}{2} = x^2 + \frac{1}{x^2}$$
$$BP_3 = 2\cos 3\frac{AP_1}{2} = x^3 + \frac{1}{x^3}$$
$$\ldots = \ldots\ldots = \ldots\ldots$$

301. **Prop.** *If the circumference of a circle* $AP_1 P_2$, *of which* O *is the centre, be divided into m equal parts in the points* $A, P_1, P_2, P_3 \ldots$, *and lines be drawn from* $P_1, P_2, P_3 \ldots$ *to any point* C *in* AO, *then*

$OA^m - OC^m = CP_1 . CP_2 . CP_3 \ldots$ *to m factors.*



For, by putting $\frac{x}{r}$ for x, in art. 183., and multiplying by r^{2m}

$$x^{2m} - 2r^m x^m + r^{2m} = \left(x^2 - 2rx\cos\frac{2\pi}{m} + r^2\right)$$
$$\left(x^2 - 2rx\cos\frac{4\pi}{m} + r^2\right)\left(x^2 - 2rx\cos\frac{6\pi}{m} + r^2\right)$$
$$\times \ldots\ldots$$

Let $AC = x$

$$\therefore (x^{2m}-2r^m x^m+r^{2m}) = (x^m-r^m)^2 = (OC^m-OA^m)^2$$

$$\left(x^2-2rx\cos\frac{2\pi}{m}+r^2\right)=OC^2-2OP_1.\,OC.\cos AOP_1+OP_1^2=CP_1^2$$

$$\left(x^2-2rx\cos\frac{4\pi}{m}+r^2\right)=OC^2-2OP_2.\,OC.\cos AOP_2+OP_2^2=CP_2^2$$

$$\left(x^2-2rx\cos\frac{6\pi}{m}+r^2\right)=OC^2-2OP_3.\,OC.\cos AOP_3+OP_3^2=CP_3^2$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots = \ldots$$

$$\therefore (OC^m-OA^m)^2=CP_1^2.\,CP_2^2.\,CP_3^2.\ldots$$

Therefore, by extracting the square root, if C be in AO,

$$OA^m-OC^m=CP_1.\,CP_2.\,CP_3\ldots.$$

If C be in OA produced

$$OC^m-OA^m=CP_1.\,CP_2.\,CP_3.\ldots.$$

Cor. In the same figure, if the arcs AP_1, P_1P_2, P_2P_3 ... be bisected in Q_1, Q_2, Q_3...

$$OA^m+OC^m=CQ_1.\,CQ_2.\,CQ_3.\ldots.$$

For, by the proposition,

$$OA^{2m}-OC^{2m}=CP_1.\,CQ_1.\,CP_2.\,CQ_2\ldots.$$

and

$$OA^m-OC^m=CP_1.\,CP_2.\,CP_3.\ldots.$$

Therefore, by division,

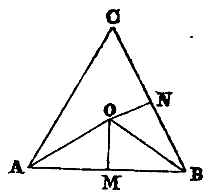
$$OA^m+OC^m=CQ_1.\,CQ_2.\,CQ_3.\ldots.$$

The two last theorems are called, from their inventor, *Cotes's Properties of the Circle.*

302. Prop. *The radius of the circle circumscribing a plane triangle is equal to* $\frac{1}{2}.\ \frac{c}{\sin C}$.

For, let ABC be the triangle. From M and N the middle points of AB, BC draw MO, NO perpendiculars to them, and

meeting in O, join AO, BO. Then (Euc. IV. 5.) O is the centre of the circle passing through A, B, C.



$$\text{Let AO} = \text{BO} = r_1 = \text{the radius}$$

$$\therefore \frac{\text{AM}}{\text{AO}} = \sin \text{AOM}$$

$$= \sin \frac{\text{AOB}}{2}$$

$$\therefore \frac{c}{2r_1} = \sin \text{C}. \quad \text{(Euc. III. 20.)}$$

$$\therefore r_1 = \frac{c}{2 \sin \text{C}}.$$

303. Cor. $\text{Sin C} = \frac{2}{ab}\sqrt{s.\,(s-a).\,(s-b).(s-c.)}$ (126.)

$$\therefore r_1 = \frac{abc}{4\sqrt{s.\,(s-a)\,(s-b).\,s-c)}}.$$

304. Prop. *The radius of a circle inscribed within a plane triangle is equal to* $\sqrt{\frac{(s-a).\,(s-b).\,(s-c)}{s}}$.

For, in the figure of the last proposition, let the angles CAB, CBA be bisected by AO, BO meeting in O. Join CO. Draw OM, ON, OP perpendiculars upon AB, BC, CA. Then (Euc. IV. 4) O is the centre of the inscribed circle, and OM=ON=OP= the radius $=r_2$, suppose. Then, since the triangle ABC is the sum of the triangles AOB, BOC, AOC,

$$\text{OM. AB} + \text{ON. BC} + \text{OP. AC} = 2\,\text{ABC}$$

$$\therefore r_2\,(a+b+c) = 2\sqrt{\{s.\,(s-a).\,(s-b).\,(s-c)\}}. \text{ (152.)}$$

$$\therefore\ r_2=\sqrt{\frac{(s-a).\ (s-b).\ (s-c)}{s}}.$$

305. Cor. 1. $r_1 . r_2=\frac{abc}{4s}=\frac{abc}{2\,(a+b+c)}$.

306. Cor. 2. Another expression for r_2 is $c.\ \dfrac{\sin\frac{A}{2}.\ \sin\frac{B}{2}}{\cos\frac{C}{2}}$.

$$\text{For } \frac{AB}{MO}=\frac{AM}{MO}+\frac{BM}{MO}$$

$$\therefore \frac{c}{r_2}=\cot\frac{A}{2}+\cot\frac{B}{2}$$

$$=\frac{\sin\left(\frac{A}{2}+\frac{B}{2}\right)}{\sin\frac{A}{2}.\ \sin\frac{B}{2}}\ (107.)$$

$$=\frac{\cos\frac{C}{2}}{\sin\frac{A}{2}.\ \sin\frac{B}{2}}$$

$$\therefore r_2=c.\ \frac{\sin\frac{A}{2}.\ \sin\frac{B}{2}}{\cos\frac{C}{2}}.$$

307. Cor. 3. $\frac{AB}{AM}=1+\frac{MB}{AM}$

$$=1+\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}$$

$$=1+\frac{s-b}{s-a}\ (125)$$

$$\therefore \frac{c}{\text{AM}} = \frac{2s-(a+b)}{s-a} = \frac{c}{s-a}$$

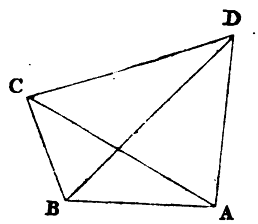
$$\therefore \text{AM} = s-a.$$

$$\text{Thus BM} = s-b,$$

$$\text{and CN} = s-c.$$

308. Prop. *If* ABCD *be a quadrilateral figure inscribed within a circle, and* a (=AB), b (=BC), c (=CD), *and* d (=DA) *be the sides,*

$$\cos \text{A} = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}$$



For, join AC, BD, then in the triangle BAD, by art. 109,

$$\text{BD}^2 = a^2 + d^2 - 2ad \cos \text{A}.$$

But in the triangle BCD

$$\text{BD}^2 = b^2 + c^2 + 2bc \cos \text{A},$$

because C = 180°—A (Euc. III. 22.)

$$\therefore 0 = -a^2 - d^2 + b^2 + c^2 + 2(ad+bc) \cos \text{A}$$

$$\therefore \cos \text{A} = \frac{a^2+d^2-b^2-c^2}{2(ad+bc)}.$$

309. Cor. 1. $\text{BD}^2 = a^2+d^2-ad.\dfrac{a^2+d^2-b^2-c^2}{ad+bc}$

$$= \frac{(a^2+d^2)\,bc + (b^2+c^2)\,ad}{ad+bc}$$

$$\therefore \text{BD} = \sqrt{\left\{\frac{(a^2+d^2)\,bc + (b^2+c^2).\,ad}{ad+bc}\right\}}$$

$$= \sqrt{\left\{\frac{(ac+bd)\ (ab+cd)}{ad+bc}\right\}}.$$

310. PROP. *The same contraction and notation being used.*

$$Sin\ A = \frac{2}{ad+bc}\sqrt{\{(s-a).\ (s-b).\ (s-c).\ (s-d)\}}$$

where $2s = a+b+c+d$.

$$\text{For } \cos A = \frac{a^2+d^2-b^2-c^2}{2\,(ad+bc)}$$

$$\therefore\ 1+\cos A = \frac{a^2+2ad+d^2-(b^2-2bc+c^2)}{2\,(ad+bc)}$$

$$= \frac{(a+d)^2-(b-c)^2}{2\,(ad+bc)}$$

$$= \frac{(a+d+b-c).\ (a+d+c-b)}{2\,(ad+bc)}.$$

Similarly,

$$1-\cos A = \frac{(a+b+c-d)\ (b+c+d-a)}{2\,(ad+bc)}.$$

Therefore, by multiplication,

$$(\sin A)^2 = \frac{2\,(s-a).\ 2\,(s-b).\ 2\,(s-c).\ 2\,(s-d)}{4\,(ad+bc)^2}$$

$$\therefore\ \sin A = \frac{2}{ad+bc}\sqrt{\{(s-a).\ (s-b).\ (s-c).\ s-d)\}}.$$

311. COR. The radius of the circumscribed circle

$$= \frac{BD}{2\sin A}\ (\text{art. } 302.)$$

$$= \tfrac{1}{4}\sqrt{\left\{\frac{(ab+cd)\ (ac+bd)\ (ad+bc)}{(s-a).(s-b).\ (s-c).\ (s-d)}\right\}}.$$

312. Prop. *The same contraction and notation being used.*

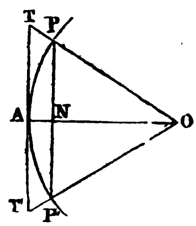
$$\textit{Area}\ ABCD = \sqrt{\{(s-a).\ (s-b).\ (s-c).\ (s-d)\}}.$$

$$\begin{aligned} \text{For } ABCD &= ABD + BCD \\ &= \tfrac{1}{2}\{ad \sin A + bc \sin C\} \\ &= \tfrac{1}{2}(ad+bc) \sin A \text{ (Euc. III. 22.)} \\ &= \sqrt{\{(s-a).(s-b).(s-c).(s-d)\}}.\text{(art. 310.)} \end{aligned}$$

313. Prop. *If a and b be sides of regular polygons of n sides inscribed within, and circumscribed about a circle, of which the radius is r, then*

$$a = 2r \sin \frac{\pi}{n}$$

$$b = 2r \tan \frac{\pi}{n}$$



For, let APP′ be the circle, O the centre, PP′ a side of the inscribed figure. Draw ONA perpendicularly to PP′: it therefore bisects PP′ and the angle POP′. (Euc. III. 3.) From A draw a tangent TAT′ meeting OP, OP′ produced in T and T′. TT′ is therefore a side of the circumscribed figure. Now the angle $PON = \frac{POP'}{2} = \frac{360°}{2n} = \frac{180°}{n}$,

$$\therefore\ a = 2\,PN = 2\,r \sin \frac{\pi}{n}$$

$$\text{and } b = 2\,AT = 2\,r \tan \frac{\pi}{n}.$$

314. Cor. 1. The perimeters are

$$n a = 2 n r \sin \frac{\pi}{n}$$

$$\text{and } n b = 2 n r \tan \frac{\pi}{n}.$$

315. Cor. 2.

$$POP' = PN.\ NO = n r^2 \sin \frac{\pi}{n}.\ \cos \frac{\pi}{n}$$

$$TOT' = AT.\ AO = r^2 \tan \frac{\pi}{n}.$$

Therefore, the areas are

$$\frac{n r^2}{2} \sin \frac{2\pi}{n} \text{ for the inscribed,}$$

$$\text{and } n r^2 \tan \frac{\pi}{n} \text{ for the circumscribed polygon.}$$

316. Prob. *To find the numerical value of the roots of a quadratic equation by aid of trigonometry.*

Let the equation be $x^2 - px + q = 0$,

or, $x^2 - px - q = 0$,

where p is positive or negative,

$$\therefore x = \frac{p}{2} \pm \frac{1}{2} \sqrt{(p^2 - 4q)},$$

$$\text{or, } = \frac{p}{2} \pm \frac{1}{2} \sqrt{(p^2 + 4q)}.$$

In the *first* case, let $4q = p^2 (\sin \alpha)^2$,

$$\therefore x = \frac{p}{2} \pm \frac{p}{2} \cos \alpha,$$

$$\left.\begin{aligned} &= p \left(\cos \frac{\alpha}{2}\right)^2 \\ \text{or, } &= p \left(\sin \frac{\alpha}{2}\right)^2 \end{aligned}\right\} \text{the two roots.}$$

In the *second* case, let $4\,q = p^2 (\tan \alpha)^2$,

$$\text{or, } p = \frac{2\sqrt{q}}{\tan \alpha}$$

$$\therefore x = \frac{p}{2} \pm \frac{p}{2} \sec \alpha$$

$$= \frac{\sqrt{q}.\cos \alpha}{\sin \alpha} . \frac{\cos \alpha. \pm 1}{\cos \alpha}$$

$$\left.\begin{aligned} &= \frac{\sqrt{q}.\left(\cos\frac{\alpha}{2}\right)^2}{\sin\frac{\alpha}{2}.\ \cos\frac{\alpha}{2}} = \sqrt{q}.\cot\frac{\alpha}{2} \\ \text{or, } &= -\frac{\sqrt{q}.\left(\sin\frac{\alpha}{2}\right)^2}{\sin\frac{\alpha}{2}.\ \cos\frac{\alpha}{2}} = -\sqrt{q}.\tan\frac{\alpha}{2} \end{aligned}\right\}\text{the two roots.}$$

317. PROB. *To solve the equation*

$$x^3 - qx + r = 0,$$

in which r is positive, or negative, and $\frac{r^2}{4} < \frac{q^3}{27}$, *and therefore all the roots real quantities.*

$$\text{Let } x = y \sin \alpha,$$

$$\therefore y^3 (\sin \alpha)^3 - q\, y \sin \alpha + r = 0,$$

$$\therefore (\sin \alpha)^3 - \frac{q}{y^2} \sin \alpha + \frac{r}{y^3} = 0.$$

But $(\sin \alpha)^3 - \frac{3}{4} \sin \alpha + \frac{1}{4} \sin 3\,\alpha = 0$, (art. 92.)

$$\therefore \frac{q}{y^2} = \frac{3}{4}$$

$$\therefore y^2 = \frac{4q}{3}$$

$$\therefore y = 2\sqrt{\frac{q}{3}}.$$

Also, $\frac{1}{4} \sin 3\,\alpha = \frac{r}{y^3}$

$$\therefore\ \sin 3\,\alpha = \frac{4r}{y^3} = \frac{r}{2} \div \left(\frac{q}{3}\right)^{\frac{3}{2}}.$$

From this equation α, and therefore $\sin \alpha$, may be found from the tables, and consequently $y \sin \alpha = 2\sqrt{\frac{q}{3}}.\ \sin \alpha$, *one* of the roots, may be determined.

But, by article 40,

$$\sin 3\,\alpha = \sin (3\,\alpha \pm 2\,\pi) = \sin (3\,\alpha \pm 4\,\pi) = \&c.$$

Therefore, the following values of $\sin \alpha$ satisfy the equation

$$\sin 3\,\alpha = \frac{r}{2} \div \left(\frac{q}{3}\right)^{\frac{3}{2}}.$$

$$\sin \alpha,\ \sin\left(\alpha \pm \frac{2\pi}{3}\right),\ \sin\left(\alpha \pm \frac{4\pi}{3}\right),\ \&c.$$

$$\text{But, } \sin\left(\alpha \pm \frac{4\pi}{3}\right) = \sin\left(\alpha \pm \frac{4\pi}{3} \mp 2\,\pi\right)\ (40.)$$

$$= \sin\left(\alpha \mp \frac{2\pi}{3}\right).$$

$$\text{Similarly, } \sin\left(\alpha \pm \frac{6\pi}{3}\right) = \sin\left(\alpha \mp \frac{4\pi}{3}\right)$$

and $\ldots\ldots = \ldots\ldots\ldots$

Therefore, the roots of the equation are

$$2\sqrt{\frac{q}{3}}.\ \sin \alpha,\ 2\sqrt{\frac{q}{3}}.\ \sin\left(\alpha + \frac{2\pi}{3}\right) \text{ and } 2\sqrt{\frac{q}{3}}.\ \sin\left(\alpha - \frac{2\pi}{3}\right).$$

318. Cor. 1. Since $\sin 3\,\alpha <$ unity, $\frac{r^2}{4} < \frac{q^3}{27}$.

Therefore, the equation can be solved by this method only when all the roots are possible. (Wood's *Algebra*, art. 331.)

319. Cor. 2. If $\frac{r^2}{4} \div \frac{q^3}{27}$ be nearly equal to unity, 3α will be nearly $\frac{\pi}{2}$, and therefore will not be determined accurately (127); a third part of this error will be entailed on α. And, in this case, $\alpha\left(=\frac{\pi}{6}\text{ nearly}\right)$ is small, and therefore a considerable error will be found in $\sin\alpha$. (98 and 127.) When this happens, $\alpha + \frac{2\pi}{3} = \pi - \frac{\pi - 3\alpha}{3}$ is nearly equal to $\pi - \frac{\pi}{6}$, and two of the roots are nearly equal.*

320. Prob. *To solve the equation*

$$x^3 + qx + r = 0,$$

where q and r are positive or negative, and $\frac{r^2}{4} + \frac{q^3}{27}$ a positive quantity, and therefore two of the roots imaginary.

$$\text{Let } x = y \sin\alpha$$

$$\therefore\ (\sin\alpha)^3 + \frac{q}{y^2}\sin\alpha + \frac{r}{y^3} = 0.$$

$$\text{But } (\sin\alpha)^3 - \frac{3}{4}\sin\alpha + \frac{1}{4}\sin 3\alpha = 0.\ (92.)$$

$$\therefore\ y^2 = -\frac{4q}{3}$$

$$\therefore\ y = 2\sqrt{-1}\sqrt{\frac{q}{3}}$$

$$\therefore\ y^3 = -8\sqrt{-1}\left(\frac{q}{3}\right)^{\frac{3}{2}}.$$

$$\text{Also, } \sin 3\alpha = \frac{4r}{y^3} = -\sqrt{-1}\frac{r}{2}\left(\frac{3}{q}\right)^{\frac{3}{2}}$$

$$\therefore\ \cos 3\alpha = \sqrt{\left(1 + \frac{27r^2}{4q^3}\right)}$$

* See the Appendix.

$$\therefore\ y^3 \cos 3\alpha = -8\sqrt{-1}\sqrt{\left(\frac{r^2}{4} + \frac{q^3}{27}\right)}$$

$$\therefore\ y^3(\cos 3\alpha + \sqrt{-1}\sin 3\alpha) = -8\sqrt{-1}\left\{-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}$$

$$\therefore\ y\ (\cos\alpha + \sqrt{-1}\sin\alpha) = 2\sqrt{-1}\ \sqrt[3]{\left\{-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}}.$$

(Art. 159.) Similarly,

$$y\ (\cos\alpha - \sqrt{-1}\sin\alpha) = -2\sqrt{-1}\sqrt[3]{\left\{-\frac{r}{2}-\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}}.$$

Therefore, by subtracting and dividing by $2\sqrt{-1}$

$$y\sin\alpha = \sqrt[3]{\left\{-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}} + \sqrt[3]{\left\{-\frac{r}{2}-\sqrt{\left(\frac{r^2}{4}+\frac{q^3}{27}\right)}\right\}}.$$

See Wood's Algebra, art. 327.

321. Cor. To find the numerical value of this quantity, firstly, suppose q *positive*.

$$\text{Let } \frac{q^3}{27} = \frac{r^2}{4}(\tan\alpha)^2$$

$$\therefore -\frac{r}{2} = -\frac{1}{\tan\alpha}\sqrt{\frac{q^3}{27}}$$

$$\therefore \sqrt[3]{-\frac{r}{2}} = -\sqrt{\frac{q}{3}}\sqrt[3]{\frac{1}{\tan\alpha}}$$

$$\therefore x = \sqrt[3]{\left\{-\frac{r}{2}+\frac{r}{2}\sec\alpha\right\}} + \sqrt[3]{\left\{-\frac{r}{2} - \frac{r}{2}\sec\alpha\right\}}$$

$$= \sqrt[3]{\left\{-\frac{r}{2}(1-\sec\alpha)\right\}} + \sqrt[3]{\left\{-\frac{r}{2}(\sec\alpha + 1)\right\}}$$

$$= \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\left(\frac{\sec\alpha + 1}{\tan\alpha}\right)} + \sqrt[3]{\left(\frac{\sec\alpha - 1}{\tan\alpha}\right)}\right\}$$

$$= \sqrt{\frac{q}{3}}\left\{\sqrt[3]{\cot\frac{\alpha}{2}} - \sqrt[3]{\tan\frac{\alpha}{2}}\right\}.$$

$$\text{Let } \sqrt[3]{\cot\frac{\alpha}{2}} = \cot\frac{\beta}{2}\ \therefore\ \sqrt[3]{\tan\frac{\alpha}{2}} = \tan\frac{\beta}{2}$$

$$\therefore\ x = \sqrt{\frac{q}{3}}\left(\cot\frac{\beta}{2} - \tan\frac{\beta}{2}\right)$$

$$= 2\sqrt{\frac{q}{3}}.\ \cot\beta.\ (88.)$$

Secondly, suppose q ***negative.***

$$\therefore x = \sqrt[3]{\left\{-\frac{r}{2}+\sqrt{\left(\frac{r^2}{4}-\frac{q^3}{27}\right)}\right\}} + \sqrt[3]{\left\{-\frac{r}{2}-\sqrt{\left(\frac{r^2}{4}-\frac{q^3}{27}\right)}\right\}}$$

$$\text{Let } \frac{q^3}{27} = \frac{r^2}{4}(\sin\alpha)^2$$

$$\text{and } \sqrt[3]{\left(\cot\frac{\alpha}{2}\right)} = \cot\frac{\beta}{2}$$

and by operating, as in the first case,

$$x = 2\sqrt{\frac{q}{3}}\ \text{cosec}\ \beta.$$

Thus is found the numerical value of one root of this cubic equation. In the second case, $\frac{r^2}{4}$ must be greater than $\frac{q^3}{27}$, for, otherwise the sine would be greater than the radius. Therefore, the remaining two roots are imaginary. (*Wood*, 331.)

322. PROP.

$$(a+b\sqrt{-1})^{\frac{1}{n}} = (a^2+b^2)^{\frac{1}{2n}}\left\{\cos\frac{1}{n}\tan^{-1}\frac{b}{a} + \sqrt{-1}\ \sin\frac{1}{n}\tan^{-1}\frac{b}{a}\right\}$$

$$\text{For } a+b\sqrt{-1} = (a^2+b^2)^{\frac{1}{2}}\left\{\frac{a}{(a^2+b^2)^{\frac{1}{2}}} + \sqrt{-1}\,\frac{b}{(a^2+b^2)^{\frac{1}{2}}}\right\}$$

$$\text{Let } \cos\alpha = \frac{a}{(a^2+b^2)}$$

$$\therefore \sin\alpha = \frac{b}{(a^2+b^2)^{\frac{1}{2}}},$$

$$\text{and } \tan\alpha = \frac{b}{a}$$

$$\therefore \alpha = \tan^{-1}\frac{b}{a}$$

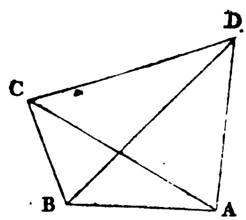
$$\therefore (a+b\sqrt{-1})^{\frac{1}{n}} = (a^2+b^2)^{\frac{1}{2n}}\ \{\cos\alpha + \sqrt{-1}\sin\alpha\}^{\frac{1}{n}}$$

$$= (a^2 + b^2)^{\frac{1}{2n}} \left\{ \cos \frac{a}{n} + \sqrt{-1} \sin \frac{a}{n} \right\}$$

$$= (a^2+b^2)^{\frac{1}{2n}} \left\{ \cos \frac{1}{n} \tan^{-1}\frac{b}{a} + \sqrt{-1} \sin \frac{1}{n} \tan^{-1}\frac{b}{a} \right\}.$$

Thus the nth root by a binomial, consisting of a real and an imaginary quantity, may be found by trigonometrical tables.

323. PROB. *If* A *and* B *be two positions, of which the distance can be measured; and* C *and* D *two objects visible from* A *and* B, *but inaccessible. To show how to find the distance of* C *and* D.



At A take the angles, BAC = A, BAD = A′.
.. B .... ABD = B, ABC = B′
Let AB = a, CBD = B″.

(I.) $\text{BC} = a.\dfrac{\sin \text{BAC}}{\sin \text{BCA}} = a\dfrac{\sin \text{A}}{\sin(\text{A}+\text{B}')}$

$= a_1$, suppose.

(II.) $\text{BD} = a.\dfrac{\sin \text{BAD}}{\sin \text{BDA}} = a\dfrac{\sin \text{A}'}{\sin(\text{A}'+\text{B})}$

$= a_2$, suppose.

(III.) $\sin \text{S} = \dfrac{2\sqrt{a_1 a_2}}{a_1+a_2} \cos \dfrac{\text{B}''}{2}$

or, $\tan \text{S}' = \dfrac{2\sqrt{a_1 a_2}}{a_1-a_2} \sin \dfrac{\text{B}''}{2}$.

(IV.) $\text{CD} = (a_1 + a_2) \cos \text{S}$,
or, $\text{CD} = (a_1 - a_2) \sec \text{S}'$.

When the distance of the objects from the stations A and B is very great compared with AB and CD, a_1 and a_2 will be nearly of the same magnitude, and B″ will be very small. In this case, the subsidiary angle S′ must be used. Whenever a_1 and a_2 are nearly of the same magnitude, and B″ nearly 180°, which will never happen except when B is near the middle point of CD, the angle S must be used.

When the four objects are in the same plane,

$$\text{B}'' = \text{B}' - \text{B}.$$

324. Prop. *In a plane triangle, of which the sides are a, b, c; and α, β, γ, the angles converted into arcs to radius unity, if α and β are very small,*

$$a = c. \frac{\alpha}{\alpha+\beta}\left\{1 + \frac{2\alpha\beta + \beta^2}{6}\right\}.$$

For $a = c. \dfrac{\sin \alpha}{\sin \gamma} = c \dfrac{\sin \alpha}{\sin (\alpha+\beta)}$.

$$\left.\begin{aligned} \text{But } \sin \alpha &= \alpha - \frac{\alpha^3}{6} \\ \sin (\alpha+\beta) &= (\alpha+\beta) - \frac{(\alpha+\beta)^3}{6} \end{aligned}\right\} (165.)$$

By neglecting the higher powers,

$$\therefore a = c. \frac{\alpha - \dfrac{\alpha^3}{6}}{(\alpha+\beta) - \dfrac{(\alpha+\beta)^3}{6}}$$

$$= c. \frac{\alpha}{\alpha+\beta}\left\{\frac{1 - \dfrac{\alpha^2}{6}}{1 - \dfrac{(\alpha+\beta)^2}{6}}\right\}$$

$$=c.\frac{\alpha}{\alpha+\beta}\left\{1-\frac{\alpha^2}{6}\right\}\left\{1+\frac{(\alpha+\beta)^2}{6}\right\}$$

$$=c.\frac{\alpha}{\alpha+\beta}\left\{1+\frac{2\alpha\beta+\beta^2}{6}\right\}.$$

325. Cor. Similarly,

$$b=c.\frac{\beta}{\alpha+\beta}\left\{1+\frac{2\alpha\beta+\alpha^2}{6}\right\}$$

$$\therefore a+b-c=\tfrac{1}{2}c\,\alpha\beta.$$

326. Prop. *If α be equal to $\pi-\theta$, where θ is very small.*

$$a=b+c-\frac{bc}{b+c}.\ \frac{\theta^2}{2}.$$

$$\begin{aligned}\text{For } a^2 &= b^2+c^2-2\,bc\cos\alpha\\ &= b^2+c^2+2\,bc\cos\theta\\ &= b^2+c^2+2\,bc\left(1-\frac{\theta^2}{2}\right)\ \ (71.)\\ &= (b+c)^2-bc\,\theta^2\\ \therefore a &= \{(b+c)^2-bc\,\theta^2\}^{\frac{1}{2}}\\ &= b+c-\frac{bc}{b+c}\,\frac{\theta^2}{2}.\end{aligned}$$

327. Prop. *If α be equal to $\pi-\theta$, when θ is very small.*

$$\beta=\frac{b\theta}{b+c}\left\{1+\frac{c(b-c)}{(b+c)^2}.\ \frac{\theta^2}{6}\right\}.$$

$$\text{For }\sin\beta=\frac{b}{a}\sin\alpha$$

$$\text{But }\sin\alpha=\sin\theta=\theta-\frac{\theta^3}{3}$$

$$\text{and }a=b+c-\frac{bc}{b+c}.\ \frac{\theta^2}{2}.$$

Therefore, by reduction,

$$\sin\beta = \frac{b\theta}{b+c}\left\{1 + \frac{bc-b^2-c}{(b+c)^2}\cdot\frac{\theta^2}{6}\right\}.$$

But $\sin\beta = \beta - \frac{\beta^3}{6}$, because, β is very small,

$$\because = \beta - \frac{(\sin\beta)^3}{6}$$

$$\therefore \beta = \sin\beta + \frac{(\sin\beta)^3}{6}$$

$$= \sin\beta\ \left\{1 + \frac{(\sin\beta)^2}{6}\right\}$$

$$= \frac{b\theta}{b+c}\left\{1 + \frac{c(b-c)}{(b+c)^2}\cdot\frac{\theta^2}{6}\right\}.$$

SECTION VII.

ON THE NUMERICAL VALUES OF THE NATURAL TRIGONOMETRIC FUNCTIONS OF ANGLES.

328. IN trigonometric tables are inserted the trigonometric functions of angles, and also the logarithms of these quantities, the former for the sake of distinction are called the *natural* trigonometric functions. These, to avoid cyphers in the first decimal places of some of the functions of small angles, are multiplied by $10000 = 10^4$. According to Def. XV. they are, therefore, in reality not functions of angles, but of arcs to radius 10^4.

The object of this section is to find the numerical value of the trigonometric functions of every angle from 0° to 90°. It will, however, be sufficient to confine the calculations to the angles from 0° to 45°, for

$$\begin{aligned}\sin(45° + A) &= \cos\{90° - (45° + A)\}\\ &= \cos(45° - A).\end{aligned}$$

Thus $\cos(45° + A) = \sin(45° - A)$,
and this will evidently obtain, in a similar manner, for the remaining trigonometric functions. Accordingly, the tables are restricted to these angles.

329. PROB. *To find sin 1′ by the continued bisection of an angle.*

First Method.

By article 75.

$$2\sin\frac{A}{2^n} = \sqrt{\{(2-\sqrt{(2+\sqrt{(2+ \ldots\ldots + \sqrt{(2+2\cos A)\}}}}}$$

Let A = 60° and n = 12,

$$\therefore \frac{A}{2^n} = \frac{60°}{2^{12}} = \frac{(60^2)'}{2^{12}} = \frac{(15^2)'}{2^8}$$

$$\therefore 2\sin\frac{(15^2)'}{2^8} = \sqrt{(2-\sqrt{(2+\sqrt{(2+ \ldots\ldots + \sqrt{3}\}}}}.$$

The symbol $\sqrt{}$ being repeated twelve times. But when the angles are very small, the sines are nearly equal to the arcs, and, therefore, proportional to the angles.

$$\therefore \frac{\sin 1'}{1} = \frac{\frac{1}{2}\sqrt{\{(2-\sqrt{(2+\sqrt{(2+ \ldots + \sqrt{3}\}}}}}{\frac{15^2}{2^8}}$$

$$\therefore \sin 1' = \frac{2^7}{15^2}\sqrt{\{(2-\sqrt{(2+\sqrt{(2+ \ldots + \sqrt{3}\}}}}.$$

Second Method.

330. $2\sin A = \sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}$

$$\therefore 2\sin 15° = \sqrt{(1+\tfrac{1}{2})} - \sqrt{(1-\tfrac{1}{2})} = 2s_1$$

$$2\sin\frac{15°}{2} = \sqrt{(1+s_1)} - \sqrt{(1-s_1)} = 2s_2$$

$$2\sin\frac{15°}{2^2} = \sqrt{(1+s_2)} - \sqrt{(1-s_2)} = 2s_3$$

$$\ldots = \ldots - \ldots = \ldots$$

$$2\sin\frac{15°}{2^{10}} = \sqrt{(1+s_{10})} - \sqrt{(1-s_{10})} = 2s_{11}$$

$$= 2\sin\frac{(15^2)'}{2^8}.$$

And sin 1′ may be formed as before.

331. Cor. 1. The latter method has the advantage of being less liable to cause errors in the result, by neglecting figures at the right of the decimal fraction.

For $2 \sin A = \surd (2 - 2 \cos 2 A)$

by differentiating, gives

$$\frac{2 \, d. \sin A}{d. \cos 2 A} = \frac{-2}{\surd (2 - 2 \cos 2 A)}.$$

But when A is small, $2 - 2 \cos 2 A$ is very small, and therefore, a small error in cos 2 A will entail a large one in sin A. But if

$$2 \sin A = \surd (1 + \sin 2 A) - \surd (1 - \sin 2 A)$$

$$\frac{2 \, d. \sin A}{d. \sin 2 A} = \frac{1}{2}\left\{\frac{1}{\surd (1 + \sin 2 A)} + \frac{1}{\surd (1 - \sin 2 A)}\right\}$$

$$= 1 \text{ nearly, when A is small.}$$

An error, therefore, in sin 2 A, will produce only half the error in sin A.

332. Prob. *Machin's series for* π.

By article 269,

$$\tan^{-1} 1 + \tan^{-1} \frac{1}{239} = \tan^{-1} \frac{1 + \frac{1}{239}}{1 - \frac{1}{239}} = \tan^{-1} \frac{120}{119}.$$

$$\text{But } \tan 4 A = \frac{4 \{\tan A - (\tan A)^3\}}{1 - 6 \tan A^2 + (\tan A)^4}$$

$$\therefore 4 \tan^{-1} \frac{1}{5} = \tan^{-1} \frac{4 \left(\frac{1}{5} - \frac{1}{5^3}\right)}{1 - \frac{6}{5^2} + \frac{1}{5^4}}$$

$$= \tan^{-1} \frac{120}{119},$$

$$\therefore \frac{\pi}{4} = 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{239}$$

$$\therefore\ \pi = 16\left\{\frac{1}{1.5} - \frac{1}{3.\,5^3} + \frac{1}{5.\,5^3} - \ldots\right\}$$
$$- 4\left\{\frac{1}{1.\,239} - \frac{1}{3.\,239^3} + \frac{1}{5.\,239^5} - \ldots\right\}$$

333. Cor. By performing the arithmetic operations,
$\pi = 3.\,14159265358979.$

334. Prob. *To find sin* 1″ *by the series for sin* a *in terms of powers of* a.

$$\text{Sin } a = a - \frac{a^3}{1.\,2.\,3} + \frac{a^5}{1.\,2.\,3.\,4.\,5} - \ldots$$

$$\text{But } a = \frac{\pi}{180^\circ}\cdot A^\circ,$$

$$\text{and } \sin a = \sin A,$$

$$\therefore\ \sin A = \frac{\pi}{180^\circ}\cdot A^\circ - \frac{1}{1.\,2.\,3}\cdot\left(\frac{\pi}{180^\circ}A^\circ\right)^3 + \ldots.$$

$$\text{Let } A^\circ = 1'' = \frac{1^\circ}{60^2},$$

$$\therefore\ \sin 1'' = \frac{\pi}{3.60^3} - \frac{1}{1.2.3}\cdot\frac{\pi^3}{3^3.\,60^9} + \frac{1}{1.2.3.4.5}\cdot\frac{\pi^5}{3^5.\,60^{15}} - \ldots$$

335. Prop.

$$\left.\begin{aligned} &\textit{Sin}\,(A+B) = 2\,\textit{sin}\,A - \textit{sin}\,(A-B) - 4\,\textit{sin}\,A.\left(\textit{sin}\,\frac{B}{2}\right)^2 \\ &\textit{cos}\,(A+B) = 2\,\textit{cos}\,A - \textit{cos}\,(A-B) - 4\,\textit{cos}\,A.\left(\textit{sin}\,\frac{B}{2}\right)^2 \end{aligned}\right\}.$$

$$\text{For } \sin(A+B) + \sin(A-B) = 2\sin A.\ \cos B$$
$$= 2\sin A\left\{1 - 2\left(\sin\frac{B}{2}\right)^2\right\}$$

$$\therefore\ \sin(A+B) = 2\sin A - \sin(A-B) - 4\sin A.\left(\sin\frac{B}{2}\right)^2$$

$$\text{and } \cos(A+B)+\cos(A-B) = 2\cos A.\ \cos B$$
$$= 2\cos A.\left\{1-2\left(\sin\frac{B}{2}\right)^2\right\}$$
$$\therefore \cos(A+B)=2\cos A-\cos(A-B)-4\cos A.\left(\sin\frac{B}{2}\right)^2.$$

336. Prob. *To calculate the sines and cosines of* $2', 3', 4' \ldots 60'$.

By substituting $(n-1)$ A for A, and A for B in art. 335.

$$\sin nA=\sin(n-1)A+\{\sin(n-1)A-\sin(n-2)A\}-4\sin(n-1)A.\left(\sin\frac{A}{2}\right)^2$$
$$\cos nA=\cos(n-1)A+\{\cos(n-1)A-\cos(n-2)A\}-4\cos(n-1)A.\left(\sin\frac{A}{2}\right)^2$$

$$\begin{aligned}
\therefore\ \sin 2' &= \sin 1' + \{\sin 1' - 0\} - 4\sin 1'(\sin 30'')^2\\
\sin 3' &= \sin 2' + \{\sin 2' - \sin 1'\} - 4\sin 2'(\sin 30'')^2\\
\sin 4' &= \sin 3' + \{\sin 3' - \sin 2'\} - 4\sin 3'(\sin 30'')^2\\
\ldots &= \ldots\ldots\ldots\ldots\\
\cos 2' &= \cos 1' + \{\cos 1' - 1\} - 4\cos 1'(\sin 30'')^2\\
\cos 3' &= \cos 2' + \{\cos 2' - \cos 1'\} - 4\cos 2'(\sin 30'')^2\\
\cos 4' &= \cos 3' + \{\cos 3' - \cos 2'\} - 4\cos 3'(\sin 30'')^2
\end{aligned}$$

337. Cor. Similarly may be found the sines and cosines of $2°, 3°, 4°$, &c.

338. Prob. *To calculate the sines and cosines of* $1°\ 1'$; $1°\ 2'$; $1°\ 3'$; . . . $1°\ 60'$ $(= 2°)$.

By making $\left.\begin{aligned} A &= 1° + (n-1)' \\ \text{and } B &= 1' \end{aligned}\right\}$ in art. 335.

$$\sin(1°+n')=\sin\{1°+(n-1)'\}+\{\sin\{1°+(n-1)'\}-\sin\{1°+(n-2)'\}\}$$
$$-4\sin\{1°+(n-1)'\}(\sin 30'')^2$$
$$\cos(1°+n')=\cos\{1°+(n-1)'\}+\{\cos\{1°+(n-1)'\}-\cos\{1°+(n-2)'\}\}$$
$$-4\cos\{1°+(n-1)'\}(\sin 30'')^2$$

$\therefore \sin 1^\circ 1' = \sin 1^\circ + \{\sin 1^\circ - \sin 59'\} - 4 \sin 1^\circ (\sin 30'')^2$
$\sin 1^\circ 2' = \sin 1^\circ 1' + \{\sin 1^\circ 1' - \sin 1^\circ\} - 4 \sin 1^\circ 1' (\sin 30'')^2$
$\sin 1^\circ 3' = \sin 1^\circ 2' + \{\sin 1^\circ 2' - \sin 1^\circ 1'\} - 4 \sin 1^\circ 1' (\sin 30'')^2$
$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

Similarly for the cosine.

339. Cor. 1. Thus may be calculated the sines and cosines of $2^\circ 1'$, $3^\circ 1'$, &c. and similarly of any angle whatever.

340 Cor. 2. Since $\tan A = \dfrac{\sin A}{\cos A}$,

$\cot A = \dfrac{\cos A}{\sin A}$, &c.=&c. all the trigonometric functions of angles may be calculated.

341. Cor. 3. It is evident, that an error in computing for any angle will affect all the succeeding results. It is necessary, therefore, to find some method of proving the correctness of the arithmetic operations. This is done by formulæ of verification, which will be explained in the following articles.

342. Prop. *Sin* $18^\circ = \dfrac{\sqrt{5}-1}{4}$.

$$\begin{aligned}
&\text{Let } A = 18^\circ\\
&3A + 2A = 90^\circ\\
\therefore\ &\cos 3A = \sin 2A\\
\therefore\ &4(\cos A)^3 - 3\cos A = 2\sin A.\cos A\\
\therefore\ &4(\cos A)^2 - 3 = 2\sin A\\
\therefore\ &(2\sin A)^2 + 2\sin A = 1\\
\therefore\ &(2\sin A)^2 + 2\sin A + \frac{1}{4} = \frac{5}{4}\\
\therefore\ &2\sin A + \frac{1}{2} = \frac{\sqrt{5}}{2}
\end{aligned}$$

$$\therefore \ \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

343. Cor. 1.

$$\text{Cos } 72^\circ = \frac{\sqrt{5}-1}{4}, \cos 36^\circ = \frac{\sqrt{5}+1}{4}. \quad (77.)$$

344. Cor. 2. $\text{Cos } 18^\circ = \sin 72^\circ = \frac{\sqrt{(10+2\sqrt{5})}}{4}.$

345. Prob. *To calculate the sines and cosines of certain angles independently of the previous methods.*

By art. 81. if A be less than 45°,

$$2 \sin A = \sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}$$
$$2 \cos A = \sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}.$$

But the sines of 18°, 30°, 45°, 60° and 72° known, (see arts. 342. and 58.) Therefore, the sines and cosines of 9°, 15°, 27° 30′ and 36° are known, consequently the sines and cosines of their complements 81°, 75°, 62° 30′ and 54°. Besides $30^\circ - 18^\circ = 12^\circ$, $45^\circ - 18^\circ = 27^\circ$, $60^\circ - 18^\circ = 42^\circ$, &c. = &c., and the sines and cosines of these angles are known. The results are to be found in a table at the end of the work.

346. Prop. *Euler's formula of verification.*

$$\sin A + \sin(72^\circ + A) - \sin(72^\circ - A) = \sin(36^\circ + A) - \sin(36^\circ - A).$$

$$\text{Sin } (70^\circ + A) - \sin (72^\circ - A) = 2 \cos 72^\circ.\ \sin A. \quad (93.)$$
$$= \frac{\sqrt{5}-1}{2} \sin A. \quad (342.)$$
$$\sin (36^\circ + A) - \sin (36^\circ - A) = 2 \cos 36^\circ.\ \sin A$$
$$= \frac{\sqrt{5}+1}{2} \sin A \quad (343.)$$

$\therefore \sin A + \sin(72° + A) - \sin(72° - A) = \sin 36° + A) - \sin(36° - A)$

By substituting any angle for A, the correctness of the sines of five angles may be investigated.

347. Cor. 1. By substituting $90° - A$ for A, and remarking, that

$$\sin(162° - A) = \sin\{180° - (162° - A)\} = \sin(18° + A)$$
$$\sin(-18° + A) = -\sin(18° - A)$$
$$\sin(126° - A) = \sin\{180° - (126° - A)\} = \sin(54° + A)$$
$$\sin(-54° + A) = -\sin(54° - A).$$

The result is

$$\sin(90° - A) + \sin(18° + A) + \sin(18° - A) = \sin(54° + A) + \sin(45° - A),$$

which is Legendre's formula of verification.

348. Cor. 2. There is another formula of the same kind,

$$\sin A = \sin(60° + A) - \sin(60° - A),$$

which is derived from the equation

$$2\cos B.\ \sin A = \sin(A + B) - \sin(B - A)$$

by making $B = 60°$.

349. Prop. *Tan* $(45° + A) - $ *tan* $(45° - A) = 2$ *tan* 2 A.

For, by articles 68. and 69.

$$\tan(45° + A) = \frac{1 + \tan A}{1 - \tan A} = \frac{\{1 + (\tan A)\}^2}{1 - (\tan A)^2}$$

$$\tan(45° - A) = \frac{1 - \tan A}{1 + \tan A} = \frac{\{1 - (\tan A)\}^2}{1 - (\tan A)^2}$$

$$\therefore \tan(45° + A) - \tan(45° - A) = \frac{4 \tan A}{1 - (\tan A)^2}$$
$$= 2 \tan 2A.$$

350. COR. $2 \tan A = \tan\left(45 + \frac{A}{2}\right) - \tan\left(45° - \frac{A}{2}\right)$

$$= \cot\left(45° - \frac{A}{2}\right) - \tan\left(45° - \frac{A}{2}\right).$$

The use of this, as a formula of verification for the tangents and cotangents, is evident.

PART II.

SPHERICAL TRIGONOMETRY.

SECTION I.

ON THE SPHERE.

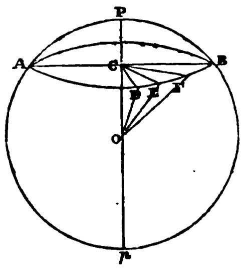
Def. I. A *sphere* is a solid bounded by a surface or superficies, of which every point is equally distant from a point within it, called the *centre*.

Def. II. The *diameter* of a sphere is any straight line which passes through the centre, and is terminated both ways by the superficies of the sphere.

1. Prop. *Every section of a sphere by a plane is a circle.*

Let AB be any section of a sphere made by a plane AB; O the centre of the sphere; from O let fall OC perpendicular to this plane; in the section AB, take any points D, E, F;

join OD, DC; OE, EC; OF, FC; then, since OC is perpendicular to the plane AB, it is perpendicular to every straight line, which meets it in that plane, (Euc. XI. Def. III.);



therefore, the angles OCD, OCE, OCF, are right angles; also, (Def. I.) OD = OE = OF; hence,

$$\begin{aligned} CD^2 &= OD^2 - OC^2, \\ &= OE^2 - OC^2, \text{ or } CE^2, \\ &= OF^2 - OC^2, \text{ or } CF^2, \\ \therefore\ CD &= CE = CF, \end{aligned}$$

or the points D, E, F, lie in the circumference of a circle, of which the centre is C, and radius = CD.

Def. III. A *great circle* is one whose plane passes through the centre of the sphere.

2. Hence, a radius of a great circle is a radius to the sphere.

Def. IV. A *small circle* is one whose plane does not pass through the centre.

3. In the above proposition, if OC = p, and radius of the sphere = r, the radius of the small circle = $(r^2 - p^2)^{\frac{1}{2}}$. Two circles are said to be parallel, when their planes are parallel; or when the line drawn from the centre of the sphere perpendicular to the plane of one of them, is also perpendicular to the plane of the other. (Euc. XI. 14.)

4. PROP. *A great circle may be drawn through any two points on the surface of a sphere, but not generally through more than two.*

For the plane of a great circle must also pass through the centre of the sphere; and a plane may be made to pass through any *three* points, but not generally more than three. (Euc. XI. 2.) Since a small circle does not pass through the centre of the sphere, it may be drawn through any three given points (not in the plane of a great circle) on the spherical superficies.

5. PROP. *Two great circles bisect each other.*

For the intersection of their planes, being a straight line passing through the centre, is a diameter of the sphere, and is therefore a diameter of both circles; which are therefore bisected. (Euc. I. Def. XVIII.)

6. PROP. *The inclination of two great circles of a sphere, is the angle made by their tangents at the point of intersection.*

For each of these tangents being perpendicular to the radius in which the planes of the circles intersect, the angle contained by the tangents measures the inclination of the planes of the circles. (Euc. XI. Def. VI.)

DEF. V. A *spherical angle* is the angle at which two arcs of great circles intersect each other on the surface of a sphere, and is the same as the angle between the tangents to the two arcs.

7. Hence, spherical angles, that are vertically opposite, are

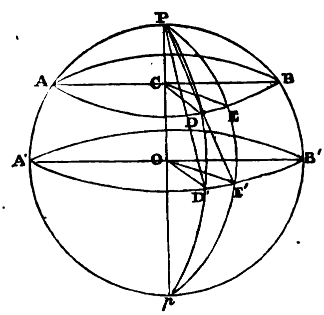
equal; as are also the angles at the opposite points of intersection of great circles. (Euc. XI. 10.)

Def. VI. If through the centre of a circle, whether great or small, a straight line be drawn perpendicular to its plane, the point in which, if produced, it meets the surface of the sphere, is called the *pole* of that circle.

8. Thus (art. 1), if OC, which is perpendicular to the plane of ABD, be produced both ways, meeting the surface of the sphere in the points P and p, these points are called the poles of ADB. It is evident that the line P Cp passes through the centre of the sphere. In a small circle, the term *pole* is more usually applied to that point only, as P, which is *nearest* to the circle. If P be called the *nearer* pole of AB, then p may be termed the *farther* pole.

9. Prop. *The pole of a circle is equally distant from every point in that circle, the distances being measured by arcs of great circles.*

The same construction remaining as in art. 1, through D and E, draw two great circles PDp, PEp, meeting the



great circle A′D′B′, whose plane is parallel to ADB, in D′ and E′; draw OD′, OE′, PD′, PE′; then, because OD′ is

equal to OE′, and that PO is common to the two triangles POD′, POE′, and at right angles to the plane A′D′B′, the two sides PO, OD′, are equal to the two sides PO, OE′, each to each, and they contain equal angles,

$\therefore$ the base PD′ = base PE′,

or the chord of the arc PD′ = chord of the arc PE′,

hence the arc PD′ = arc PE′. (Euc. III. 28.)

In the same manner, it is shown that the distance of P from all other points in the circle A′D′B′ is measured by equal arcs of great circles, or P is the pole of A′D′B′.

10. Cor. 1. The distance of P from every point in the circle ADB is also measured by equal arcs of great circles, therefore P is the pole of ADB.

11. Cor. 2. Because PO = OD′, PD′ is the chord of a quadrantal arc; the distance, therefore, of every point of a great circle from its pole is a quadrant of a great circle.

12. Cor. 3. Since a line, which is perpendicular to two lines meeting it in a plane, is perpendicular to that plane, if a point P can be found such that its distance, measured by a great circle, from each of the two points D′ and E′ not in the same diameter, is a quadrant, that point is the pole of the great circle passing through D′ and E′.

13. Prop. *The angles* PD′E′ *and* PE′D′ *are right angles.*

Since PO is perpendicular to D′OE′, the plane POD′ is perpendicular to the plane D′OE′, (Euc. XI. 18.); and the angle PD′E′ is therefore a right angle, by (art. 6.) Similarly the angle P′ED′ is a right angle.

14. Cor. The four spherical angles at D′ or E′ are equal to four right angles; and since these angles comprehend *all* the angular space at D′ or E′, it follows that all the spherical angles, formed by the intersections of ever so many arcs of great circles at a point on the surface of the sphere, are equal to four right angles.

15. Prop. *The tangent of* DB *at* D *is perpendicular to the tangent of* DP.

For the tangent of D′B′ at D′ is perpendicular to the tangent of D′P; and it is also perpendicular to D′O, therefore it is perpendicular to the plane D′OP, (Euc. XI. 4.); as is also the tangent of DB at D, which is parallel to it, (Euc. XI. 8.); therefore the tangent of DB at D is perpendicular to the tangent of DE.

16. Prop. *If* a = *the arc of a great circle subtending a spherical angle at its pole, the spherical angle* $= \frac{180}{\pi} a$; *radius of the sphere being* = 1.

Let the spherical angle = A, then it is obvious that A is as often contained in 180°, as a is in π the semi-circumference of a circle whose rad = 1, or that,

$$\frac{A}{a} = \frac{180^\circ}{\pi}$$

$$\text{or } A = \frac{180^\circ}{\pi} . a.$$

17. Cor. 1. If A = 90°, then $a = \frac{\pi}{2}$; that is the quadrant of a great circle subtends at its pole a spherical angle = 90°.

18. Cor. 2. If a' be an arc subtending the same angle rad $= r$, then $a = \frac{a'}{r}$.

$$\therefore A = \frac{180^\circ}{\pi} \cdot \frac{a'}{r},$$

and $\frac{a}{a'} = \frac{1}{r}$, or $\frac{D'E'}{DE} = \frac{1}{CD} = \frac{1}{\sin PD} = \frac{1}{\cos DD'}$.

19. Cor. 3. It has been shown, in (art. 19. pt. I.), that the plane angle subtended by an arc $(= a) = \frac{180^\circ}{\pi}$, a; hence the spherical angle is equal to the corresponding plane angle at the centre of a great circle, at the pole of which the spherical angle is formed: thus, in the last figure, the angle D′PE′ is equal to the angle D′OE′.

20. Cor. 4. Since all the plane angles at O are equal to four right angles, all the spherical angles that can be formed at P, are likewise equal to four right angles, which has indeed been already shown.

Def. VII. A *lune* is that portion of the surface of a sphere, which is included between two great semicircles described on the same diameter.

21. Prop. *To find the area of a lune.*

In the preceding figure, let PDpE be the lune $=$ L, A′D′B′ the great circle, whose poles are P and p, and $a =$ D′E′; then it is plain that L is contained as often in half the surface of the sphere ($= \frac{S}{2}$, suppose), as a is contained in A′D′B′ $= \pi$;

$$\therefore \frac{L}{a} = \frac{\frac{1}{2}S}{\pi},$$

$$= \frac{S}{2\pi},$$

$$\text{or } L = \frac{S}{2\pi} \cdot a;$$

$$\text{but } S = 4\pi, \text{ rad} = 1,$$

$$\therefore \frac{S}{2\pi} = 2,$$

and $L = 2a$, and $\propto a$, the arc of the lune;

or, $L = \frac{\pi}{180^\circ} . 2A$, and $\propto A$, the angle of the lune.

22. Cor. 1. If $A = 90^\circ$,

$$L = \pi,$$

and there are four such lunes on the whole surface of the sphere.

23. Cor. 2. The great circle, of which P and p are the poles, divides each of these four equal and rectangular lunes into two equal portions, each of which $= \frac{\pi}{2}$, or $\frac{\pi r^2}{2}$, rad$=r$.

24. Cor. 3. If the radius of the sphere $= r$:

$$L = \frac{4\pi r^2}{2\pi r} . ra',$$

$$= 2r^2 a';$$

$$\text{and if } a' = \frac{\pi r}{2},$$

$$L = \pi r^2.$$

$$\text{Also, } L = \frac{\pi r^2}{180^\circ} . 2A.$$

Def. VIII. The *axis* of a sphere is a diameter perpendicular to the plane of a great circle.

Thus POp is the axis of the sphere corresponding to the great circle $A'D'B'$.

25. PROP. *If a be the distance between the poles of two great circles, the inclination of their planes* $= \dfrac{180^\circ}{\pi}. a$.

Since the axes of great circles are perpendicular to their planes, they are inclined at the same angle as the planes of the circles; hence the arc which measures the distance of the poles is equal to the arc which measures the inclination of the planes of the great circles; which inclination is therefore $= \dfrac{180^\circ}{\pi}. a$.

26. COR. If the supplemental arc be taken, the angle under the planes is obtuse, and $= \dfrac{180^\circ}{\pi}. (\pi - a)$.

DEF. IX. A *plane* is a *tangent* to a sphere, when their surfaces have only one point in common.

27. PROB. *To draw a tangent plane to any point on the surface of a sphere.*

Let any two arcs of great circles pass through the point to which the tangent plane is to be drawn, and let tangents to these arcs be drawn; join the centre of the sphere and the proposed point; let any plane pass through one of the tangents, and let it be turned about this tangent until it pass through any point assumed in the other tangent; then, because the proposed point and the assumed point are in this plane, the tangent, in which these points are, is in this plane (Euc. I.

Def. VII.); but the radius of the sphere stands at right angles to the tangents of the two arcs at the point of their intersection, it is therefore (Euc. XI. 4.) at right angles to the plane in which the tangents are; and since the tangents of the arcs at their intersection touch the sphere only in one point, the plane passing through them also, touches the sphere only in one point; it is therefore a tangent plane.

SECTION II.

ON SPHERICAL TRIANGLES.

Def. X. A *spherical triangle* is a portion of the surface of a sphere contained by three arcs of great circles; and is formed by taking any three points, (of which two are in the plane of a great circle,) on the surface of the sphere, and letting three arcs of great circles pass through them, intersecting each other.

28. Let it be stated, once for all, that in the following pages, the Greek letters α, β, γ, are used to denote the arcs which form spherical triangles to rad $= 1$, unless the contrary be expressed, and that A, B, C, are the spherical angles opposite to the sides α, β, γ.

29. Prop. *Any two sides of a spherical triangle taken together are greater than the third.*

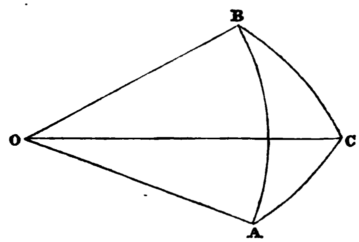
Let ABC be any spherical triangle; O the centre of the sphere; join OA, OB, OC.

Then the angle BOC $= \dfrac{180^\circ}{\pi}. \alpha$, (art. 19. pt. I.)

. . . . AOC $= \dfrac{180^\circ}{\pi}. \beta$,

$$AOB = \frac{180^\circ}{\pi} . \gamma,$$

but the angles BOC, AOC, are greater than the angle AOB, (Euc. XI. 20.)



$$\therefore \frac{180^\circ}{\pi} (\alpha + \beta) \text{ is greater than } \frac{180^\circ}{\pi} . \gamma;$$

that is, $\alpha + \beta$ is greater than γ:
similarly, $\alpha + \gamma$ . . . . . β,
$\beta + \gamma$ . . . . . α.

30. Prop. *The sum of the three sides of a spherical triangle is less than the circumference of a great circle of the sphere.*

By (Euc. XI. 21.) the sum of the plane angles at O is less than four right angles; therefore, from the preceding article,

$$\frac{180^\circ}{\pi} (\alpha + \beta + \gamma) \text{ is less than } 2.\ 180^\circ,$$

$$\text{or } \alpha + \beta + \gamma \quad . \quad . \quad . \quad . \quad 2\pi;$$

that is, the sum of the three sides is less than the circumference of a great circle, rad $= 1$.

31. Cor. If α', β', γ', be arcs to rad $= r$,

$$\text{then } \alpha = \frac{\alpha'}{\gamma}, \beta = \frac{\beta'}{\gamma}, \gamma = \frac{\gamma'}{\gamma},$$

$$\text{hence } \frac{1}{r}(\alpha' + \beta' + \gamma') \text{ is less than } 2\pi,$$

$$\text{or } \alpha' + \beta' + \gamma' \quad . \quad . \quad . \quad . \quad 2\pi r.$$

Def. XI. A *spherical polygon* is a portion of the surface of a sphere contained by several arcs of great circles.

32. **Prop.** *The sum of the sides of a spherical polygon is less than the circumference of a great circle.*

Let α, β, γ, δ, &c. be the sides of the polygon subtending plane angles at the centre of the sphere; then, since the sum of these plane angles is less than four right angles, (Euc. XI. 21.)

$$\frac{180°}{\pi}(\alpha + \beta + \gamma + \delta, \&c.) \text{ is less than } 2.\ 180°,$$

$$\text{or } \alpha + \beta + \gamma + \delta, \&c. \quad . \quad . \quad . \quad . \quad 2\pi.$$

33. **Cor.** To rad $= r$, $\alpha' + \beta' + \gamma' + \delta'$, &c. is less than $2\pi r$.

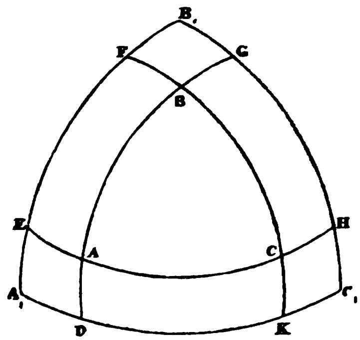
Def. XII. If with the angular points of any spherical triangle, as poles, three great circles be described intersecting each other, they will form another spherical triangle, which is called the *polar*, or *supplemental* triangle.

The proposed triangle, for the sake of distinction, will be called the *primary triangle.*

34. **Prop.** *The angular points of the polar triangle are the poles of the sides of the primary triangle.*

Let $A_1B_1C_1$ be the polar triangle described according to the definition: then, because A is the pole of B_1C_1, the distance

of A from every point in B_1C_1 is measured by the arc of a quadrant, (art. 11.); therefore the distance of A from C_1 is a



quadrant. Similarly, the distance of B from C_1 is a quadrant; hence C_1 is the pole of AB; in the same manner it is shown that A_1 and B_1 are the poles of BC, and AC.

35. Prop. *If* A, B, C, *and* A_1, B_1, C_1, *be the angles of the primary and polar triangles, opposite to the sides denoted by* α, β, γ, *and* α_1, β_1, γ_1: *then,*

$$A = \frac{180^\circ}{\pi}(\pi - \alpha_1),$$

$$B = \frac{180^\circ}{\pi}(\pi - \beta_1),$$

$$C = \frac{180^\circ}{\pi}(\pi - \gamma_1);$$

also, $$A_1 = \frac{180^\circ}{\pi}(\pi - \alpha),$$

$$B_1 = \frac{180^\circ}{\pi}(\pi - \beta),$$

$$C_1 = \frac{180^\circ}{\pi}(\pi - \gamma).$$

Produce the sides of ABC, if necessary, to meet the sides of the polar triangle in D, E, F, &c., then,

$$GH = C_1G - C_1H = C_1G - (B_1C_1 - B_1H),$$
$$= \frac{\pi}{2} - \left(a_1 - \frac{\pi}{2}\right), \text{ (art. 34.)}$$
$$= \pi - a_1,$$

but $\dfrac{GH}{\pi} = \dfrac{A}{180^\circ}$, (art. 16.)

$$\therefore GH = \frac{A}{180^\circ}, \pi = \pi - a_1,$$

hence $A = \dfrac{180^\circ}{\pi}(\pi - a_1)$;

similarly, $B = \dfrac{180^\circ}{\pi}(\pi - \beta_1)$,

$$C = \frac{180^\circ}{\pi}(\pi - \gamma_1).$$

Again, since $BC = CF - BF$

$$= CF - (FK - BK),$$
$$= \frac{\pi}{2} - FK + \frac{\pi}{2},$$
$$\therefore a = \pi - FK,$$

and $FK = \pi - a$,

but, as before, $\dfrac{FK}{\pi} = \dfrac{A_1}{180^\circ}$,

$$\therefore A_1 = \frac{180^\circ}{\pi}(\pi - a);$$

similarly, $B_1 = \dfrac{180^\circ}{\pi}(\pi - \beta)$,

$$C_1 = \frac{180^\circ}{\pi}(\pi - \gamma).$$

36. COR. 1. Since, by (art. 27. pt. I.), $f A = f a$, where by (art. 19. pt. I.) $A = \dfrac{180^\circ}{\pi} . a$; it follows, that in the above equations between the sides and angles of the primary and polar triangles,

$$\sin A = \sin(\pi - a_1),$$
$$\sin A_1 = \sin(\pi - a),$$

and so on for the other trigonometric functions.

37. Cor. 2. $A = \dfrac{180°}{\pi}(\pi - a_1) = 180° - \dfrac{180°}{\pi}.\, a_1$

$$= 180° - A_1, \text{ since } \frac{180°}{\pi}.\, a_1 = A_1, \text{ (16.)}$$

$$\therefore A + A_1 = 180°.$$

Similarly, $B + B_1 = 180°$,

$$C + C_1 = 180°.$$

38. Cor. 3. The arc subtending the spherical angle $A + a_1 = \pi$,

$$\text{that is, } GH + a_1 = \pi;$$
$$\text{similarly, } DK + \beta_1 = \pi,$$
$$EF + \gamma_1 = \pi.$$

39. Cor. 4. The arc subtending the spherical angle $A_1 + a = \pi$,

$$\text{that is, } FK + a = \pi,$$
$$\text{similarly, } HE + \beta = \pi,$$
$$DG + \gamma = \pi.$$

40. Cor. 5. If any one of the angles as $A = 90°$; then

$$90° = \frac{180°}{\pi}(\pi - a_1),$$
$$= 180° - \frac{180°}{\pi}.\, a_1,$$
$$\therefore 1 = 2 - \frac{2\, a_1}{\pi},$$
$$\text{and } a_1 = \frac{\pi}{2} = \text{quadrantal arc.}$$

Also, from (art. 37), $A_1 = 90°$,

$$\text{and } \therefore a = \frac{\pi}{2}.$$

41. Cor. 6. If $A = B = C = 90°$,

$$\text{then } a_1 = \beta_1 = \gamma_1 = \frac{\pi}{2},$$
$$A_1 = B_1 = C_1 = 90°;$$
$$a_1 = \beta_1 = \gamma_1 = \frac{\pi}{2};$$

and the primary triangle coincides with the polar.

42. Cor. 7. If $B = C = 90°$, and $A = 180° - D_2$,

$$\text{then } D_2 = \frac{180°}{\pi}(a_1 + \beta_1 + \gamma_1 - \pi),$$
$$\therefore a_1 + \beta_1 + \gamma_1 = \pi + \text{arc which subtends } D_2,$$
$$\text{and } \gamma_1 = \text{arc which subtends } D_2.$$

43. Prop. *The sum of the three angles of a spherical triangle is always less than six right angles, and greater than two.*

(1.) For, from (art. 35.),

$$A + B + C = \frac{180°}{\pi}\{3\pi - (a_1 + \beta_1 + \gamma_1)\},$$
$$= 6.\ 90° - \frac{180°}{\pi}(a_1 + \beta_1 + \gamma_1),$$
$$= 6.\ 90° - \text{an angle};$$

since the sides of the polar triangle must have some magnitude.

(2.) Since $\alpha_1 + \beta_1 + \gamma_1 = 2\pi -$ an arc, (art. 30.)

$$= 2\pi - x, \text{ suppose,}$$

$$\therefore A + B + C = 6.90^\circ - \frac{180^\circ}{\pi}(2\pi - x),$$

$$= 6.90^\circ - 4.90^\circ + \frac{180^\circ}{\pi}x,$$

$$= 2.90^\circ + \frac{180^\circ}{\pi}x,$$

$$= 2.90^\circ + \text{an angle}.$$

44. Cor. 1. Hence, a spherical triangle may have two or three right angles; or two or three obtuse angles.

45. Cor. 2. If $S = \frac{A + B + C}{2}$,

$$S = 90^\circ + \frac{\text{an angle}}{2};$$

$$\text{also, } S = 3.90^\circ - \frac{\text{an angle}}{2};$$

hence, cos S is always negative.

46. Prop. *The difference between any two angles of a spherical triangle and the third angle $= 2.90^\circ -$ an angle.*

For, from (art. 35.),

$$A + B - C = \frac{180^\circ}{\pi}\{\pi - (\alpha_1 + \beta_1 - \gamma_1)\},$$

$$= 2.90^\circ - \frac{180^\circ}{\pi}(\alpha_1 + \beta_1 - \gamma_1),$$

$$= 2.90^\circ - \text{an angle};$$

in the same manner it is shown, that $A + C - B$, and $B + C - A$, are each less than 180°.

47. Cor. Since $S = \dfrac{A+B+C}{2}$, (art. 45.)

$$S - C = \frac{A+B-C}{2},$$

$$= 90^\circ - \frac{\text{an angle}}{2},$$

$\therefore$ cos (S — C) is always positive,

cos (S — B) . . . .

cos (S — A) . . . .

48. Similar equations to those, which have been shown to subsist between the sides and angles of the primary and polar triangles, exist between the sides and angles of what may be termed the primary and polar polygons; each of the sides of the primary polygon being less than a semicircle, since the sum of the sides of any spherical polygon is less than the circumference of a great circle, (art. 32.)

49. Prop. *If two spherical triangles have two sides of the one equal to two sides of the other, each to each; and have likewise the angles contained by those sides equal to one another, the two triangles shall be equal.*

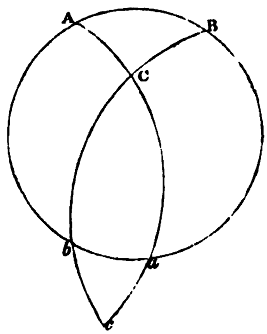
This proposition is proved by precisely the same reasoning as is used in (Euc. I. 4.)

50. Prop. *The area of a spherical triangle*

$= \dfrac{\pi}{180^\circ}(A + B + C - 180^\circ)$, *rad* $= 1$; *to rad* $= r$, *the area*

$= \dfrac{\pi r^2}{180^\circ}(A + B + C - 180^\circ)$.

Let the sides AC and BC be produced, cutting the great circle of which AB is a portion in a and b, and intersecting each other in c:



$$\text{then, } \mathrm{AC}a = \mathrm{C}ac = \pi,$$

$$\therefore\ \mathrm{AC} = ac,$$

$$\text{similarly, } \mathrm{BC} = bc;$$

also, the angle at C $= c$, because these angles measure the inclination of the same planes; therefore, by the preceding proposition, the triangles ABC, abc, are equal to each other.

Let a = area of the triangle, then by (art. 21.),

$$\mathrm{AC}a\mathrm{B} = \frac{2\pi}{180^\circ}.\ \mathrm{A} = a + \mathrm{BC}a,$$

$$\mathrm{BA}b\mathrm{C} = \frac{2\pi}{180^\circ}.\ \mathrm{B} = a + \mathrm{AC}b,$$

$$\mathrm{C}\,bca = \frac{2\pi}{180^\circ}.\ \mathrm{C} = a + b\mathrm{C}a;$$

hence, by addition, (since $\frac{1}{2}$ surface of sphere

$$= \text{area of two great circles},$$

$$= 2\,\pi = a + \mathrm{BC}a + \mathrm{AC}b + b\mathrm{C}a,)$$

$$2a + 2\,\pi = \frac{2\pi}{180^\circ}\ (\mathrm{A} + \mathrm{B} + \mathrm{C}),$$

$$\therefore 2a = \frac{2\pi}{180^\circ}(A + B + C) - 2\pi,$$

$$= \frac{2\pi}{180^\circ}(A + B + C - 180^\circ),$$

$$\therefore a = \frac{\pi}{180^\circ}(A + B + C - 180^\circ);$$

to rad $= r$,

$$a = \frac{\pi r^2}{180^\circ}(A + B + C - 180^\circ), \quad \text{(art. 24.)}$$

51. Cor. 1. Since the limits of $(A + B + C)$ are (art. 43.) 2.90° and 6.90°,

$A + B + C - 180^\circ =$ any number of degrees between 0 and 4.90°.

52. Cor. 2. If $A + B + C = 4.90^\circ$,

$$a = \pi r^2.$$

53. Cor. 3. If $A = B = C = 90^\circ$,

$$a = \frac{\pi r^2}{2}, \text{ as in (art. 23.)}$$

54. Cor. 4. If A' be the angle of a lune whose area $= a$, since

$$\frac{2\pi A'}{180^\circ} = \frac{\pi}{180^\circ}(A + B + C - 180^\circ),$$

$$A' = \frac{A + B + C}{2} - 90^\circ.$$

55. Cor. 5. Since $\frac{\pi}{180^\circ} = \frac{\text{arc of } 1''}{1''} = \frac{\sin 1''}{1''}$,

$$= \frac{1}{57^\circ\, 17'\, 44''.\, 772}, \text{ (art. 18. pt. I.)}$$

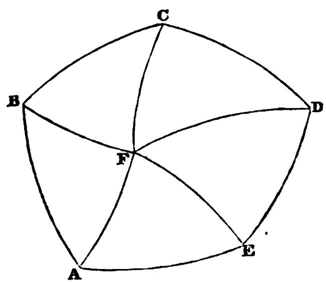
$\therefore$ 57° 17′ 44″. 772 sin 1″ = 1″;

hence the area of a spherical triangle expressed in seconds $= (A + B + C - 180°)\, r^2 \sin 1''$, where the number of seconds is found by finding how often (A + B + C − 180°) contains 47° 17′ 44″. 772 (= 57°. 29577 = 206264″. 772.)

56. Cor. 6. $A + B + C - 180° = \dfrac{180°}{\pi}\, a$, which is generally expressed by the letter E, and is called the *Spherical Excess.*

57. Prop. *The area of a spherical polygon of n sides* $= \dfrac{\pi}{180°}$. {*sum of its angles* − $(n - 2)$ 180°}.

Let ABCDE be any polygon, F any point in its surface,



through F, and the angular points A, B, C, &c., draw arcs FA, FB, &c. of great circles; then, it is evident, the polygon will be divided into as many triangles as it has sides, that is, into n triangles, the area of which $= \dfrac{\pi}{180°}$(angles of polygon + angles at F − n. 180°), but the angles at F = 2. 180°, (art. 14.), and the area of the triangles = area of polygon $= \dfrac{\pi}{180°}$ {angles of polygon − $(n - 2)$ 180°}.

58. Cor. If rad $= r$; this expression must be multiplied by r^2.

Def. XIII. A *right-angled spherical triangle* is that which has at least *one* of its angles a right angle. A *quadrantal triangle* is that which has at least *one* of its sides equal to a quadrant. All other spherical triangles are called *oblique-angled triangles.*

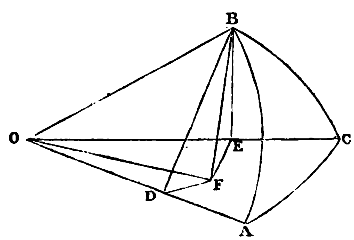
SECTION III.

ON THE INVESTIGATION OF SUCH EQUATIONS AND PROPERTIES AS ARE NECESSARY FOR THE SOLUTION OF SPHERICAL TRIANGLES.

59. PROP. *In any triangle ;*

$$\frac{\sin A}{\sin B} = \frac{\sin \alpha}{\sin \beta}, \quad \frac{\sin A}{\sin C} = \frac{\sin \alpha}{\sin \gamma}, \quad \frac{\sin B}{\sin C} = \frac{\sin \beta}{\sin \gamma}.$$

Let O be the centre of the sphere; join OA, OB, OC; in the planes BOA, BOC, from C let fall the perpendiculars



BD, BE, to OA, OC, respectively; also let fall BF perpendicular to the plane AOC, and join DF, FE, OF; then,

$$BO^2 = OF^2 + FB^2 = OE^2 + EB^2,$$
$$= OE^2 + FB^2 + FE^2,$$
$$\therefore OF^2 = OE^2 + FE^2,$$

hence (Euc. I. 48.) the angle OEF is a right angle, and the

angle BEF (Euc. XI. Def. VI.) is the inclination of the planes AOC, BOC, that is, the angle BEF $=$ C. Similarly, it may be shown that the angle BDF $=$ A.

$$\text{Now } \frac{BF}{BE} = \sin C, \text{ (art. 30. pt. I.)}$$

$$\text{and } \frac{BE}{BO} = \sin \alpha, \text{ (art. 24. pt. I.)}$$

$$\therefore \frac{BF}{BO} = \sin C.\ \sin \alpha : (1.)$$

$$\text{also } \frac{BF}{BD} = \sin A,$$

$$\text{and } \frac{BD}{BO} = \sin \gamma,$$

$$\therefore \frac{BF}{BO} = \sin A.\ \sin \gamma : (2.)$$

hence, equating (1.) and (2.), and dividing

$$\frac{\sin A}{\sin C} = \frac{\sin \alpha}{\sin \gamma}.$$

Similarly, it is shown that,

$$\frac{\sin A}{\sin B} = \frac{\sin \alpha}{\sin \beta}, \text{ and } \frac{\sin B}{\sin C} = \frac{\sin \beta}{\sin \gamma}.$$

60. Cor. If $C = 90°$; $\sin A = \dfrac{\sin \alpha}{\sin \gamma}$, and $\sin B = \dfrac{\sin \beta}{\sin \gamma}$. And, if $\alpha = \beta = \gamma$, $\sin A = \sin B = \sin C$.

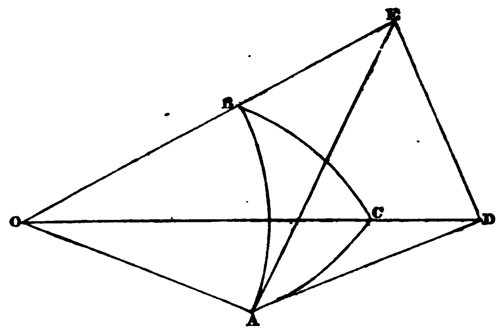
61. Prop. *In any triangle;*

$$\cos A = \frac{\cos \alpha - \cos \beta.\ \cos \gamma}{\sin \beta.\ \sin \gamma},$$

$$\cos B = \frac{\cos \beta - \cos \alpha.\ \cos \gamma}{\sin \alpha.\ \sin \gamma},$$

$$\cos C = \frac{\cos \gamma - \cos \alpha.\ \cos \beta}{\sin \alpha.\ \sin \beta}.$$

In fig. (art. 59.) draw AD, AE tangents to the arcs AC, AB, at A; produce OC, OB, to meet the tangents in D and E; join AE, DE.



Then, $AD = \tan\beta$, $OD = \sec\beta$;

$AE = \tan\gamma$, $OE = \sec\gamma$;

and $DE^2 = AE^2 + AD^2 - 2AE.\ AD.\cos DAE$, (I.; 109.)

$= (\tan\beta)^2 + (\tan\gamma)^2 - 2\tan\beta.\ \tan\gamma.\ \cos A$; (1.)

also $DE^2 = OE^2 + OD^2 - 2\ OE.\ OD.\cos BOC$,

$= (\sec\beta)^2 + (\sec\gamma)^2 - 2\sec\beta.\ \sec\gamma.\ \cos\alpha$,

$= 1 + (\tan\beta)^2 + 1 + (\tan\gamma)^2 - 2\sec\beta.\ \sec\gamma.\ \cos\alpha$; (2.)

hence, by subtracting (1.) from (2.),

$$2 + 2\tan\beta.\ \tan\gamma.\ \cos A - 2\sec\beta.\ \sec\gamma.\ \cos\alpha = 0,$$

$$\therefore \cos A = \left(\frac{\cos\alpha}{\cos\beta.\ \cos\gamma} - 1\right).\ \left(\frac{\cos\beta.\ \cos\gamma}{\sin\beta.\ \sin\gamma}\right),$$

$$= \frac{\cos\alpha - \cos\beta.\ \cos\gamma}{\sin\beta.\ \sin\gamma}.$$

Similarly, $\cos B = \dfrac{\cos\beta - \cos\alpha.\ \cos\gamma}{\sin\alpha.\ \sin\gamma}$,

$$\cos C = \frac{\cos\gamma - \cos\alpha.\ \cos\beta}{\sin\alpha.\ \sin\gamma}.$$

62. Cor. 1. If $C = 90°$, $\cos C = 0$,

$$\therefore \cos\gamma = \cos\alpha.\ \cos\beta.$$

Since the cosine of an arc less than a quadrant is positive, and of an arc greater than a quadrant, negative; this equation shows that α and β, (the sides which contain the right angle), must be either *both less* or *both greater* than a quadrant, when $\cos \gamma$ is positive; that is, when γ is *less than* a quadrant: when *one* of the sides about the right angle is *greater* than a quadrant, $\cos \gamma$ is negative, and γ is therefore *greater than* a quadrant.

63. Cor. 2. If the radius of the sphere $= r$,

$$\cos A = \frac{r \cos \alpha' - \cos \beta'. \cos \gamma'}{\sin \beta'. \sin \gamma'}.$$

64. Prop. *The angles at the base of an isosceles spherical triangle are equal to one another.*

Let the side $\alpha =$ side β,

$$\text{then } \cos A = \frac{\cos \alpha - \cos \beta. \cos \gamma}{\sin \beta. \sin \gamma}, \text{ (art. 61.)}$$

$$= \frac{\cos \alpha}{\sin \alpha}. \frac{1 - \cos \gamma}{\sin \gamma},$$

$$= \cot \alpha. \tan \frac{\gamma}{2};$$

$$\text{similarly, } \cos B = \cot \alpha. \tan \frac{\gamma}{2},$$

$$\therefore \cos A = \cos B,$$

$$\text{and } A = B.$$

65. Cor. If $\alpha = \beta = \gamma$,

then $A = B = C$;

that is, if the triangle be equilateral it is also equiangular.

$$1-(\cos A)^2 = \frac{4}{(\sin\beta . \sin\gamma)^2} \sin\sigma . \sin(\sigma-\alpha) . \sin(\sigma-\beta) . \sin(\sigma-\gamma),$$

$$\therefore \sin A = \frac{2}{\sin\beta . \sin\gamma} \{\sin\sigma . \sin(\sigma-\alpha) . \sin(\sigma-\beta) . \sin(\sigma-\gamma)\}^{\frac{1}{2}}.$$

The four equations just established obtain also for the other two angles B and C.

69. Cor. If $\alpha = \beta$, $\sin A = \sin B$,

$$\text{and } \sin A = \left\{ \sin\left(\alpha + \frac{\gamma}{2}\right) . \sin\left(\alpha - \frac{\gamma}{2}\right)\right\}^{\frac{1}{2}};$$

if $\alpha = \beta = \gamma$,

$$\sin A = \sin B = \sin C = \frac{1}{2\left(\cos\frac{\alpha}{2}\right)^2}\left\{3-4\left(\sin\frac{\alpha}{2}\right)^2\right\}^{\frac{1}{2}}.$$

70. Prop.

$$Sin\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}} . \cos\frac{C}{2}, \quad (1.)$$

$$\cos\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\frac{\gamma}{2}} . \sin\frac{C}{2}, \quad (2.)$$

$$\sin\left(\frac{A-B}{2}\right) = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\frac{\gamma}{2}} . \cos\frac{C}{2}, \quad (3.)$$

$$\cos\left(\frac{A-B}{2}\right) = \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\frac{\gamma}{2}} . \sin\frac{C}{2}. \quad (4.)$$

(1.) For $\sin\left(\frac{A+B}{2}\right) = \sin\frac{A}{2} . \cos\frac{B}{2} . + \cos\frac{A}{2} . \cos\frac{B}{2}$,

(art. 64. pt. I.)

hence, by substituting for the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, &c., as found in (art. 68.),

$$\sin\left(\frac{A+B}{2}\right)=\left(\frac{\sin(\sigma-\beta).\ \sin(\sigma-\gamma)}{\sin\beta.\ \sin\gamma}\right)^{\frac{1}{2}}.\ \left(\frac{\sin\sigma.\ \sin(\sigma-\beta)}{\sin\alpha.\ \sin\gamma}\right)^{\frac{1}{2}}$$

$$+\left(\frac{\sin\sigma.\ \sin(\sigma-\alpha)}{\sin\beta.\ \sin\gamma}\right)^{\frac{1}{2}}.\ \left(\frac{\sin(\sigma-\alpha).\ \sin(\sigma-\beta)}{\sin\alpha.\ \sin\gamma}\right)^{\frac{1}{2}},$$

$$=\frac{1}{\sin\gamma}.\ \{\sin(\sigma-\beta)+\sin(\sigma-\alpha)\}.\left(\frac{\sin\sigma.\ \sin(\sigma-\gamma)}{\sin\alpha.\ \sin\beta}\right)^{\frac{1}{2}},$$

$$=\frac{1}{\sin\gamma}.\ 2\sin\left(\sigma-\frac{\alpha+\beta}{2}\right).\ \cos\left(\frac{\alpha-\beta}{2}\right).\ \cos\frac{C}{2},$$

$$=\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}}.\ \cos\frac{C}{2}.$$

(2.) Again, by expanding $\cos\left(\frac{A+B}{2}\right)$ by (art. 66. pt. I.) and for $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, &c., it is found that,

$$\cos\frac{A+B}{2}=\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\frac{\gamma}{2}}.\ \sin\frac{C}{2}.$$

The equations (3.) and (4.) are investigated in the same way as (1.) and (2.), which will be used hereafter in proving *Lhuillier's* Theorem.

71. PROP.

$$Tan\left(\frac{A+B}{2}\right)=\frac{cos\left(\frac{\alpha-\beta}{2}\right)}{cos\left(\frac{\alpha+\beta}{2}\right)}.\ cot\frac{C}{2},$$

$$tan\left(\frac{A-B}{2}\right)=\frac{sin\left(\frac{\alpha-\beta}{2}\right)}{sin\left(\frac{\alpha+\beta}{2}\right)}.\ cot\frac{C}{2}.$$

From (art. 70.), dividing (1.) by (2.),

$$\frac{\sin\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} = \tan\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} \cdot \cot\frac{C}{2};$$

again, from the same article, by dividing (3.) by (4.),

$$\frac{\sin\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} = \tan\left(\frac{A-B}{2}\right) = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)} \cdot \cot\frac{C}{2}.$$

The equations of this proposition are usually called *Napier's* first and second Analogies.

72. Cor. 1. If $C = 90°$, in a right-angled triangle,

$$\tan\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)},$$

$$\text{and } \tan\frac{A-B}{2} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}.$$

73. Cor. 2.

$$\operatorname{Cot}\frac{C}{2} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \cdot \tan\left(\frac{A+B}{2}\right) = \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)} \cdot \tan\left(\frac{A-B}{2}\right),$$

$$\text{and } \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} = \frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)},$$

in any triangle.

74. Prop. *In any triangle the greater angle is opposite to the greater side.*

For, from equation (3.) in (art. 70.),

$$\sin\left(\frac{A-B}{2}\right)=\frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\frac{\gamma}{2}}.\cos\frac{C}{2};$$

and $\cos\frac{C}{2}$, and $\sin\frac{\gamma}{2}$ are both positive, therefore $\sin\left(\frac{A-B}{2}\right)$ and $\sin\left(\frac{\alpha-\beta}{2}\right)$ must have the same sign; that is, if A be greater or less than B, α must be greater or less than β, which consideration proves the truth of the proposition.

75. Similarly, the equation;

$$\cos\left(\frac{A+B}{2}\right)=\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\frac{\gamma}{2}}.\sin\frac{C}{2},$$

shows that $\frac{A+B}{2}$ and $\frac{\alpha+\beta}{2}$ have the same sign; that is, if $\frac{A+B}{2}$ be greater or less than 90°, then $\frac{\alpha+\beta}{2}$ is greater or less than $\frac{\pi}{2}$.

76. Prop. *In any triangle;*

$$\cos\alpha=\frac{\cos A+\cos B.\cos C}{\sin B.\sin C},$$

$$\cos\beta=\frac{\cos B+\cos A.\cos C}{\sin A.\sin C},$$

$$\cos\gamma=\frac{\cos C+\cos A.\cos B}{\sin A.\sin B}.$$

Let α_1, β_1, γ_1, be the sides
A_1, B_1, C_1, the opp. angles } of the polar triangle;

$$\text{then, } \cos A_1 = \frac{\cos \alpha_1 - \cos \beta_1 . \cos \gamma_1}{\sin \beta_1 . \sin \gamma_1}, \text{ (art. 61.)}$$

$$\text{and } A_1 = \frac{180^\circ}{\pi}(\pi - \alpha), \text{ (art. 35.)}$$

$$A = \frac{180^\circ}{\pi}(\pi - \alpha_1),$$

$$\&c. = \&c.$$

$$\text{hence, } \cos A_1 = - \cos \alpha, \text{ (art. 46. pt. I.)}$$

$$\text{and } \cos A = - \cos \alpha_1;$$

$$\text{similarly, } \cos B = - \cos \beta_1,$$

$$\cos C = - \cos \gamma_1;$$

$$\text{also, } \sin A = \sin \alpha_1,$$

$$\sin B = \sin \beta_1,$$

$$\sin C = \sin \gamma_1;$$

hence, by effecting the requisite substitutions,

$$\cos \alpha = \frac{\cos A + \cos B . \cos C}{\sin B . \sin C}.$$

In the same manner, $\cos \beta$ and $\cos \gamma$ are expressed in terms of the angles.

77. Cor. 1. If $A = B$,

$$\cos \alpha = \frac{\cos A}{\sin A} . \left(\frac{1 + \cos C}{\sin C}\right),$$

$$= \cot A . \cot \frac{C}{2} = \cos \beta.$$

This shows that if two angles of a spherical triangle be equal to each other, the sides which are opposite to these equal angles are likewise equal. In this case,

$$\cos \gamma = \cos C . (\operatorname{cosec} A)^2 + (\cot A)^2.$$

78. Cor. 2. If $A = B = C$,

$$\cos a = \cot A.\cot\frac{A}{2} = \cos\beta = \cos\gamma,$$

$$\therefore a = \beta = \gamma.$$

79. Cor. 3. If $C = 90°$, then $\cos C = 0$,

$$\text{and } \cos a = \frac{\cos A}{\sin B};$$

in this equation, the sign of $\cos a$ depends on the sign of $\cos A$, which is positive whilst A is less than 90°, and negative when A is greater than 90°; hence, in a right-angled spherical triangle, if A be greater or less than 90°, a is greater or less than a quadrant.

80. Prop. *In any triangle,*

$$\textit{if } S = \frac{A+B+C}{2};$$

$$\sin\frac{a}{2} = \left(\frac{-\cos S.\cos(S-A)}{\sin B.\sin C}\right)^{\frac{1}{2}},$$

$$\cos\frac{a}{2} = \left(\frac{\cos(S-B).\cos(S-C)}{\sin B.\sin C}\right)^{\frac{1}{2}},$$

$$\tan\frac{a}{2} = \left(\frac{-\cos S.\cos(S-A)}{\cos(S-B).\cos(S-C)}\right)^{\frac{1}{2}},$$

$$\sin a = \frac{2}{\sin B.\sin C}\left(-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)\right)^{\frac{1}{2}}.$$

(1.) From (art. 76.),

$$\cos a = \frac{\cos A + \cos B.\cos C}{\sin B.\sin C},$$

$$\therefore 1 - \cos a = -\frac{\cos A - (\cos B.\cos C - \sin B.\sin C)}{\sin B.\sin C},$$

$$= -\frac{\cos A + \cos(B+C)}{\sin B.\sin C},$$

$$= -\frac{2\cos\left(\frac{A+B+C}{2}\right).\ \cos\left(\frac{B+C-A}{2}\right)}{\sin B.\ \sin C}, \text{ (art. 96. pt. I.)}$$

$$= -\frac{2\cos S.\ \cos(S-A)}{\sin B.\ \sin C}. \quad (1.)$$

since $\frac{B+C-A}{2} = S - A$,

hence, $\sin\frac{a}{2} = \left(-\frac{\cos S.\ \cos(S-A)}{\sin B.\ \sin C}\right)^{\frac{1}{2}}$. (a.)

(2.) $1 + \cos a = \frac{\cos A + \cos(B-C)}{\sin B.\ \sin C}$,

$$= \frac{2\cos\left(\frac{A+B-C}{2}\right).\ \cos\left(\frac{A+C-B}{2}\right)}{\sin B.\ \sin C},$$

$$= \frac{2\cos(S-B).\ \cos(S-C)}{\sin B.\ \sin C}, \quad (2.)$$

hence, $\cos\frac{a}{2} = \left(\frac{\cos(S-B).\ \cos(S-C)}{\sin B.\ \sin C}\right)^{\frac{1}{2}}$. (b.)

(3.) By dividing (a.) by (b.),

$$\tan\frac{a}{2} = \left(\frac{-\cos S.\ \cos(S-A)}{\cos(S-B).\ \cos(S-C)}\right)^{\frac{1}{2}}.$$

(4.) By multiplying (1.) and (2.),

$$1 - (\cos a)^2 = (\sin a)^2,$$

$$= -\frac{4}{\sin B.\ \sin C}\cos S.\cos(S-A).\ \cos(S-B).\cos(S-C),$$

$$\therefore \sin a = \frac{2}{\sin B.\ \sin C}\{-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)\}^{\frac{1}{2}}.$$

Similarly,

$$\sin\beta = \frac{2}{\sin A.\sin C}\{-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)\}^{\frac{1}{2}},$$

$$\sin\gamma = \frac{2}{\sin A.\sin B}\{-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)\}.$$

In (art. 45.) it was shown that cos S is always negative; and in (art. 47.), that cos (S—A), &c. is always positive; therefore, the above equations are always possible.

81. PROP. *In any triangle;*

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} \cdot \tan\frac{\gamma}{2},$$

$$\tan\left(\frac{\alpha-\beta}{2}\right) = \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} \cdot \tan\frac{\gamma}{2}.$$

From the polar triangle, and (art. 71.),

$$\tan\left(\frac{A_1+B_1}{2}\right) = \frac{\cos\left(\frac{a_1-\beta_1}{2}\right)}{\cos\left(\frac{a_1+\beta_1}{2}\right)} \cdot \cot\frac{C_1}{2},$$

$$\tan\left(\frac{A_1-B_1}{2}\right) = \frac{\sin\left(\frac{a_1-\beta_1}{2}\right)}{\sin\left(\frac{a_1+\beta_1}{2}\right)} \cdot \cot\frac{C_1}{2};$$

and, from the equations in (art. 35.),

$$\frac{A_1+B_1}{2} = \frac{180^\circ}{\pi}\left(\pi - \frac{\alpha+\beta}{2}\right) = 180^\circ - \frac{180^\circ}{\pi}\left(\frac{\alpha+\beta}{2}\right),$$

$$\therefore\ 180^\circ - \frac{A_1+B_1}{2} = \frac{180^\circ}{\pi}\left(\frac{\alpha+\beta}{2}\right),$$

$$\text{and } \tan\left(\frac{A_1+B_1}{2}\right) = -\tan\left(\frac{\alpha+\beta}{2}\right);$$

$$\text{similarly, } \tan\left(\frac{A_1-B_1}{2}\right) = -\tan\left(\frac{\alpha-\beta}{2}\right);$$

$$\text{also, } \frac{180^\circ}{\pi}\left(\frac{\alpha_1+\beta_1}{2}\right) = 180^\circ - \frac{A+B}{2},$$

$$\therefore \cos\frac{\alpha_1+\beta_1}{2} = -\cos\left(\frac{A+B}{2}\right),$$

$$\text{and } \sin\left(\frac{\alpha_1 \pm \beta_1}{2}\right) = \pm\sin\left(\frac{A \pm B}{2}\right);$$

$$\text{again, } \cos\left(\frac{\alpha_1-\beta_1}{2}\right) = \cos\left(\frac{A-B}{2}\right),$$

$$\text{and } \cot\frac{C_1}{2} = \tan\frac{\gamma}{2}:$$

hence, by substitution,

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)}\cdot\tan\frac{\gamma}{2}, \quad (1.)$$

$$\text{and } \tan\left(\frac{\alpha-\beta}{2}\right) = \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)}\cdot\tan\frac{\gamma}{2}. \quad (2.)$$

The equations (1.) and (2.) constitute *Napier's* second and fourth Analogies.

82. Cor. 1.

If $\frac{\gamma}{2} = \frac{\pi}{4}$, $\tan\frac{\gamma}{2} = 1$, in the equations (1.) and (2.)

83. Cor. 2.

$$\operatorname{Tan}\frac{\gamma}{2} = \frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}\cdot\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\left(\frac{A+B}{2}\right)}{\sin\left(\frac{A-B}{2}\right)}\cdot\tan\left(\frac{\alpha-\beta}{2}\right),$$

$$\text{and } \frac{\tan\left(\frac{\alpha+\beta}{2}\right)}{\tan\left(\frac{\alpha-\beta}{2}\right)} = \frac{\tan\left(\frac{\mathrm{A}+\mathrm{B}}{2}\right)}{\tan\left(\frac{\mathrm{A}-\mathrm{B}}{2}\right)}.$$

84. Prop. *In a spherical triangle;*

$$\cot \mathrm{A} = \cot \alpha.\ \sin \beta.\ \operatorname{cosec} \mathrm{C} - \cos \beta.\ \cot \mathrm{C},$$
$$\text{and } \cot \alpha = \cot \mathrm{A}.\ \sin \mathrm{C}.\ \operatorname{cosec} \beta + \cos \mathrm{C}.\ \cot \beta.$$

$$\operatorname{Cos} \mathrm{A} = \frac{\cos \alpha - \cos \beta.\ \cos \gamma}{\sin \beta.\ \sin \gamma}, \text{ by (art. 76.)}$$

$$\therefore \cos \mathrm{A}.\ \sin \beta.\ \sin \gamma = \cos \alpha - \cos \beta.\ \cos \gamma,$$
$$= \cos \alpha - \cos \beta.\ (\cos \alpha.\ \cos \beta + \sin \alpha.\ \sin \beta.\ \cos \mathrm{C})$$
$$= \cos \alpha\ (\sin \beta)^2 - \sin \alpha.\ \sin \beta.\ \cos \beta.\ \cos \mathrm{C},$$

and $\sin \gamma = \sin \alpha.\ \dfrac{\sin \mathrm{C}}{\sin \mathrm{A}}$, by (art. 59.)

$\therefore \cot \mathrm{A}.\ \sin \alpha.\ \sin \mathrm{C} = \cos \alpha.\ \sin \beta - \sin \alpha.\ \cos \beta.\ \cos \mathrm{C}$, ($a$.)

hence, $\cot \mathrm{A} = \cot \alpha.\ \sin \beta.\ \operatorname{cosec} \mathrm{C} - \cos \beta.\ \cot \mathrm{C}$; (1.)

also, from (a.), $\cos \alpha.\ \sin \beta = \cot \mathrm{A}.\ \sin \alpha.\ \sin \mathrm{C} + \sin \alpha.\ \cos \beta.\ \cos \mathrm{C}$,

$\therefore \cot \alpha = \cot \mathrm{A}.\ \sin \mathrm{C}.\ \operatorname{cosec} \beta + \cos \mathrm{C}.\ \cot \beta$. (2.)

85. Cor. If, instead of taking the angle C, and the side β, the angle B and side γ be taken, the equations (1.) and (2.) become

$$\cot \mathrm{A} = \cot \alpha.\ \sin \gamma.\ \operatorname{cosec} \mathrm{B} - \cos \gamma.\ \cot \mathrm{B},$$
$$\text{and } \cot \alpha = \cot \mathrm{A}.\ \sin \mathrm{B}.\ \operatorname{cosec} \gamma + \cos \mathrm{B}.\ \cot \gamma.$$

The equation (1.) is chiefly useful for finding the corresponding small variations of the parts of a spherical triangle.

SECTION IV.

ON THE APPLICATION OF THE EQUATIONS ESTABLISHED IN THE PRECEDING SECTION TO THE SOLUTION OF SPHERICAL TRIANGLES.

I. *On the solution of triangles, which have only one right angle.*

86. Since one of the three angles $= 90°$, by the hypothesis, it follows that there are two angles and three sides, or five quantities in a right-angled triangle, which must be attended to in the following investigation. From the equations between the sides and angles, in (art. 59. and 61.), it appears that a certain number of equations arise expressed in three of the above-mentioned five parts; but five things, combined three and three together, produce ten combinations, therefore there are ten equations requisite for the solution of all the cases of the triangle under consideration. It is obvious that these ten cases are included in the following six, in which the specified parts are given; viz.

(1.) The hypothenuse and adjacent angle.
(2.) The hypothenuse and adjacent side.
(3.) The two sides about the right angle.

(4.) One of the sides containing the right angle, and the opposite angle.

(5.) One of the sides containing the right angle, and the adjacent angle.

(6.) The two angles.

87. For common purposes, a *technical* memory has been invented, under the title of *Napier's* Rules for Circular Parts, which are described as follows :

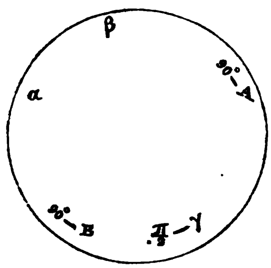
The five circular parts are,

the two sides,

the complement of the hypothenuse,

and the complements of the two angles

Any one of these is called a *middle part ;* the two next to it are then called the *adjacent parts,* and the two remaining ones the *opposite parts.* Thus, writing in order the five parts in the circumference of a circle, if α be taken as the *middle*



part, β and $90° - B$ are the *adjacent parts,* and $90° - A$ and $\frac{\pi}{2} - \gamma$ are the *opposite parts ;* similarly for any other part. The two rules to be established are,

(1.) *The sine of the middle part = product of tangents of adjacent parts.*

(2.) *The sine of the middle part = product of cosines of opposite parts.*

88. *To prove Napier's Rules.*

$$\text{From (art. 61.), } \cos C = \frac{\cos\gamma - \cos\alpha.\cos\beta}{\sin\alpha.\sin\beta},$$

$$\text{let } C = 90^\circ,$$

$$\therefore \cos\gamma = \cos\alpha.\cos\beta; \qquad (1.)$$

$$\text{and } \cos A = \frac{\cos\alpha - \cos\beta.\cos\gamma}{\sin\beta.\sin\gamma},$$

$$= \frac{\dfrac{\cos\gamma}{\cos\beta} - \cos\beta.\cos\gamma}{\sin\beta.\sin\gamma},$$

$$= \frac{\cos\gamma}{\sin\gamma}.\frac{\sin\beta}{\cos\beta} = \tan\beta.\cot\gamma; \qquad (2.)$$

$$\text{similarly, } \cos B = \tan\alpha.\cot\gamma. \qquad (3.)$$

$$\text{From (art. 59.), } \frac{\sin A}{\sin C} = \frac{\sin\alpha}{\sin\gamma},$$

$$\therefore \sin\alpha = \sin A.\sin\gamma, \qquad (4.)$$

$$\text{similarly, } \sin\beta = \sin B.\sin\gamma. \qquad (5.)$$

From (art. 61. and 59.),

$$\text{since } C = 90^\circ,$$

$$\cot A \sin\alpha = \cos\alpha.\sin\beta,$$

$$\therefore \sin\beta = \tan\alpha.\cot A; \qquad (6.)$$

$$\text{similarly, } \sin\alpha = \tan\beta.\cot B. \qquad (7.)$$

From (2.) and (4.),

$$\text{since } \cos A = \frac{\cos\gamma}{\sin\gamma}.\frac{\sin\beta}{\cos\beta},$$

$$\text{and } \sin B = \frac{\sin\beta}{\sin\gamma},$$

$$\therefore \frac{\cos A}{\sin B} = \frac{\cos\gamma}{\cos\beta} = \cos\alpha,$$

$$\text{hence, } \cos A = \cos\alpha.\sin B; \qquad (8.)$$

$$\text{similarly, } \cos B = \cos\beta.\sin A. \qquad (9.)$$

Lastly, from (6.) and (7.),

$$\cot A. \cot B = \cos a. \cos \beta,$$

$$\therefore \cos \gamma = \cot A. \cot B. \quad (10.)$$

In these equations, if the proper substitutions be made for the circular parts, it will be observed that *Napier's* rules are established, and that in a form adapted to logarithmic computation, since they are readily reduced to radius $= r$, the logarithm of which is 10.

89. It now remains to apply *Napier's* rules to the six cases already stated.

CASE I. *Given* γ *and* A, *to find the remaining parts.*

The rules stated in (art. 87.) immediately give,

$$\sin a = \sin \gamma. \sin A,$$
$$\tan \beta = \tan \gamma. \cos A,$$
$$\cot B = \cos \gamma. \tan A.$$

If the angle B be given instead of A;

$$\sin \beta = \sin \gamma. \sin B,$$
$$\tan a = \tan \gamma. \cos B,$$
$$\cot A = \cos \gamma. \tan B.$$

CASE II. *Given* γ *and the side* a.

$$\text{Then } \sin A = \frac{\sin a}{\sin \gamma},$$
$$\cos B = \tan a. \cot \gamma,$$
$$\cos \beta = \frac{\cos \gamma}{\cos a}.$$

The ambiguity of sin A is removed by remembering (art. 79.) that if a be greater or less than $\frac{\pi}{2}$, A must also be greater or less than 90°.

Similarly, γ and the side β being given;

$$\sin B = \frac{\sin \beta}{\sin \gamma},$$

$$\cos A = \tan \beta . \cot \gamma,$$

$$\cos a = \frac{\cos \gamma}{\cos \beta}.$$

CASE III. *Given* a *and* β.

$$\text{Cos } \gamma = \cos a . \cos \beta,$$

$$\tan A = \frac{\tan a}{\sin \beta},$$

$$\tan B = \frac{\tan \beta}{\sin a}.$$

CASE IV. *Given* A *and* a.

$$\text{Sin } B = \frac{\cos A}{\cos a},$$

$$\sin \beta = \tan a . \cot A,$$

$$\sin \gamma = \frac{\sin a}{\sin A}.$$

Since the sine of an arc or angle is equal to the sine of the supplemental arc or angle, the values of these sines are doubtful; and by producing the arcs, or sides β and γ, until they intersect each other on the surface of the sphere, the angle formed at their intersection will $=$ A, and will also be opposite to a; so that there will be two right-angled triangles having the same data. This is the only ambiguous case in the solution of right-angled triangles.

Similarly, if B and β be given;

$$\sin A = \frac{\cos B}{\cos \beta},$$

$$\sin \alpha = \tan \beta . \cot B,$$

$$\sin \gamma = \frac{\sin \beta}{\sin B}.$$

CASE V. *Given* α *and the adjacent angle* B.

$$\text{Then } \cos A = \sin B . \cos \alpha,$$

$$\tan \beta = \sin \alpha . \tan B,$$

$$\tan \gamma = \frac{\tan \alpha}{\cos B}.$$

Similarly, if β and A be given;

$$\cos B = \sin A . \cos \beta,$$

$$\tan \alpha = \sin \beta . \tan A,$$

$$\tan \gamma = \frac{\tan \beta}{\cos A}.$$

CASE VI. *Given the two angles* A *and* B.

And by the rules,

$$\cos \gamma = \cot A . \cot B,$$

$$\cos \beta = \frac{\cos B}{\sin A},$$

$$\cos \alpha = \frac{\cos A}{\sin B}.$$

90. These rules are to be applied in subserviency to the considerations already made in (art. 127. pt. I.). Thus, if α and γ be given to determine B, since

$$\cos B = \tan \alpha . \cot \gamma;$$

if B be small, or nearly 180°, it cannot be determined by *Napier's* rules with sufficient accuracy; the difficulty is to be avoided in the following manner:

$$\left(\tan \frac{B}{2}\right)^2 = \frac{1 - \cos B}{1 + \cos B}, \text{ (art. 77. pt. I.)}$$

$$=\frac{1-\tan\alpha.\ \cot\gamma}{1+\tan\alpha.\ \cot\gamma},$$

$$=\frac{\cos\alpha.\ \sin\gamma-\sin\alpha.\ \cos\gamma}{\cos\alpha.\ \sin\gamma+\sin\alpha.\ \cos\gamma},$$

$$=\frac{\sin(\gamma-\alpha)}{\sin\ \gamma+\alpha)},$$

$$\therefore\ \tan\frac{B}{2}=\sqrt{\left(\frac{\sin(\gamma-\alpha)}{\sin(\gamma+\alpha)}\right)};$$

$\tan\dfrac{B}{2}$ is necessarily positive, and the value of B will be found from this equation with sufficient accuracy.

91. Similarly, if A and B be given to find γ, which is small, or nearly π, it must be found from the equation

$$\left(\tan\frac{\gamma}{2}\right)^2=-\frac{\cos(A+B)}{\cos(A-B)};$$

and not from

$$\cos\gamma=\cot A.\ \cot B.$$

92. If $\alpha=\dfrac{\pi}{2}$ nearly, it cannot be found accurately from

$$\sin\alpha=\sin\gamma.\ \sin A;$$

but since $\left\{\tan\left(\dfrac{\pi}{4}-\dfrac{\alpha}{2}\right)\right\}^2=\dfrac{1-\sin\alpha}{1+\sin\alpha}$, (art. 80. pt. I.)

$$=\frac{1-\sin\gamma.\ \sin A}{1+\sin\gamma.\ \sin A};$$

$$=\frac{1-\tan\phi}{1+\tan\phi},\ \text{suppose},$$

$$=\tan\left(\frac{\pi}{4}-\phi\right),$$

$$\therefore\ \tan\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)=\sqrt{\left\{\tan\left(\frac{\pi}{4}-\phi\right)\right\}}:$$

hence $\dfrac{\pi}{4}-\dfrac{\alpha}{2}$, and therefore α is accurately found.

93. It must be remembered in the solution of right-angled triangles, that $\alpha + \beta$ is greater or less than π, according as $A + B$ is greater or less than $180°$; and that $A + B$ necessarily exceeds one right angle.

II. *On the solution of quadrantal triangles.*

94. Let C_1 be the angle of the polar triangle corresponding to $\gamma = \frac{\pi}{2}$ in the proposed triangle,

$$\text{then } C_1 = \frac{180°}{\pi}(\pi - \gamma), \quad \text{by (art. 34.)}$$
$$= \frac{180°}{\pi}\left(\pi - \frac{\pi}{2}\right),$$
$$= 90°;$$

that is, the polar triangle has a right angle; also, any two parts of the primary triangle being given, the two corresponding parts of the polar triangle are also given; hence the polar triangle being solved by *Napier's* rules, the sides and angles of the quadrantal triangle become known from the equations in (art. 88.). But *Napier's* rules may be immediately applied to the quadrantal triangle by making the two angles adjacent to the quadrantal side, the complements of the two other sides, and the complement of the angle opposite to the quadrantal arc, the circular parts: thus, by making α_1 the middle part,

$$\sin \alpha_1 = \tan \beta_1 . \tan (90° - B_1),$$

$$\text{where } \frac{180°}{\pi}\alpha_1 = 180° - A, \text{ and } \therefore \sin \alpha_1 = \sin A,$$

$$\frac{180°}{\pi}\beta_1 = 180° - B, \text{ and } \therefore \tan \beta_1 = -\tan B,$$

$$\text{and } \tan (90° - B_1) = -\cot\left(\frac{180°}{\pi}\beta\right),$$

$$= -\tan\left(90^\circ - \frac{180^\circ}{\pi}\beta\right),$$

$$= -\tan\frac{180^\circ}{\pi}\left(\frac{\pi}{2} - \beta\right),$$

$$= -\tan\left(\frac{\pi}{2} - \beta\right),$$

hence, by substitution,

$$\sin A = \tan B . \tan\left(\frac{\pi}{2} - \beta\right),$$

similarly, $\sin B = \tan A . \tan\left(\frac{\pi}{2} - \alpha\right)$:

in the same way it may be shown, that *Napier's* rules are true for the other eight cases.

III. *On the solution of oblique-angled triangles.*

95. From (art. 59. 61. and 76.) it appears that if any three parts of an oblique-angled triangle be given, a fourth part may be found; hence, since there are six parts of the triangle in question, namely, the three sides, and three angles, it follows that there will be in all *fifteen* cases; because six things combined four and four together produce fifteen combinations; but these fifteen cases are evidently contained in the following six, in which are specified the parts given to find the remaining, that is:

(1.) The three sides.
(2.) The three angles.
(3.) Two sides and the included angle.
(4.) Two angles and the included side.
(5.) Two sides and the angle opposite to one of them.
(6.) Two angles and the side opposite to one of them.

96. Case I. *Given the three sides.*

It has been shown in (art. 68.), that

$$\sin\frac{A}{2} = \left(\frac{\sin(\sigma-\beta).\ \sin(\sigma-\gamma)}{\sin\beta.\ \sin\gamma}\right)^{\frac{1}{2}}, \quad (1.)$$

$$\cos\frac{A}{2} = \left(\frac{\sin\sigma.\ \sin(\sigma-\alpha)}{\sin\beta.\ \sin\gamma}\right)^{\frac{1}{2}}, \quad (2.)$$

$$\tan\frac{A}{2} = \left(\frac{\sin(\sigma-\beta).\ \sin(\sigma-\gamma)}{\sin\sigma.\ \sin(\sigma-\alpha)}\right)^{\frac{1}{2}}, \quad (3.)$$

$$\sin A = \frac{2}{\sin\beta.\ \sin\gamma}\{\sin\sigma.\ \sin(\sigma-\alpha).\ \sin(\sigma-\beta).\sin\sigma-\gamma\}^{\frac{1}{2}}. \quad (4.)$$

When $\frac{A}{2}$ is nearly 90°, or when A is nearly 180°, the equation (1.) does not give a result sufficiently accurate; because, to a given increment of the arc, the corresponding increment of the sine is then the least. (Art. 127. pt. I.)

When A is small, the variation of the cosine is likewise small, therefore the equation (2.) cannot be used with advantage in this case.

Again, if $\frac{A}{2}$ be nearly 90°, or A be nearly 180°, and the calculations are made by the common tables, the third equation must not be used; because, when the angle is nearly equal to 90°, the increment of the tangent, which varies as the secant, is then very great (art. 127. pt. I.); which circumstance causes an inaccuracy in the proportional parts of the tables; but since A is seldom nearly 180°, the equations (1.) and (3.) may be generally used.

If only one angle be sought, the shortest solution is derived from the three first equations; but if all the angles be required, the last equation affords a solution as brief and convenient as the three former.

97. The angle A may also be found by means of a *subsidiary* arc: for,

$$\cos A = \frac{\cos \alpha - \cos \beta . \cos \gamma}{\sin \beta . \sin \gamma}, \quad \text{(art. 61.)}$$

let θ be this subsidiary arc, corresponding to the subsidiary angle $\frac{180^\circ}{\pi} \theta$, and let

$$\cos \theta = \cos \beta . \cos \gamma,$$

where θ is the hypothenuse of a right-angled triangle, of which sides equal to β and γ contain the right angle; consequently θ will be greater or less than $\frac{\pi}{2}$, according to what has been observed in (art. 62.),

$$\begin{aligned} \text{hence, } \cos A &= \frac{\cos \alpha - \cos \theta}{\sin \beta . \sin \gamma}, \\ &= -\frac{(\cos \theta - \cos \alpha)}{\sin \beta . \sin \gamma}, \\ &= \frac{2 \sin \left(\frac{\theta + \alpha}{2}\right) . \sin \left(\frac{\theta - \alpha}{2}\right)}{\sin \beta . \sin \gamma}, \quad \text{(art. 96. pt. I.)} \end{aligned}$$

98. If the arc of a great circle be drawn from the angle A perpendicular to the side α, the triangle may be solved by *Napier's* rules in the following manner:

Let χ and ψ be the segments of α made by the arc $(= \epsilon)$ perpendicular to it;

$$\begin{aligned} \text{then } \cos \beta &= \cos \chi . \cos \epsilon, \\ \cos \gamma &= \cos \psi . \cos \epsilon, \\ \therefore \frac{\cos \beta}{\cos \gamma} &= \frac{\cos \chi}{\cos \psi}, \end{aligned}$$

$$\text{and } \frac{\cos \beta \pm \cos \gamma}{\cos \gamma} = \frac{\cos \chi \pm \cos \psi}{\cos \psi},$$

$$\therefore \frac{\cos \beta - \cos \gamma}{\cos \beta + \cos \gamma} = \frac{\cos \chi - \cos \psi}{\cos \chi + \cos \psi},$$

$$\text{and } \tan\left(\frac{\gamma+\beta}{2}\right).\tan\left(\frac{\gamma-\beta}{2}\right) = \tan\left(\frac{\psi+\chi}{2}\right).\tan\left(\frac{\psi-\chi}{2}\right),$$

$$= \tan\frac{a}{2}.\tan\left(\frac{\psi-\chi}{2}\right),$$

$$\text{hence } \tan\frac{\psi-\chi}{2} = \cot\frac{a}{2}.\tan\left(\frac{\gamma+\beta}{2}\right).\tan\left(\frac{\gamma-\beta}{2}\right),$$

$\therefore \tan\left(\frac{\psi-\chi}{2}\right)$, and consequently $\frac{\psi-\chi}{2}$ is found;

Let $\psi - \chi = \delta$,
and $\psi + \chi = a$,

$$\therefore \psi = \frac{a+\delta}{2}, \text{ and } \chi = \frac{a-\delta}{2}.$$

Hence, if ψ be the segment adjacent to B,

$$\cos B = \cot\gamma.\tan\left(\frac{a+\delta}{2}\right);$$

$$\text{similarly, } \cos C = \cot\beta.\tan\left(\frac{a-\delta}{2}\right).$$

$$\text{Again, since } \cos\epsilon.\cos\frac{a+\delta}{2} = \cos\gamma,$$

$$\therefore \cos\epsilon = \frac{\cos\gamma}{\cos\left(\frac{a+\delta}{2}\right)},$$

$$\text{or } \cos\epsilon = \frac{\cos\beta}{\cos\left(\frac{a-\delta}{2}\right)}.$$

99. If the perpendicular fall *without* the base, then $\psi-\chi = a$, and $\frac{\psi+\chi}{2}$ must be found in the same manner as $\frac{\psi-\chi}{2}$ has been found above.

100. CASE II. *Given the three angles.*

As in the preceding case, from (art. 80.)

$$\sin\frac{a}{2} = \left(\frac{-\cos S.\ \cos(S-A)}{\sin B.\ \sin C}\right)^{\frac{1}{2}}, \quad (1.)$$

$$\cos\frac{a}{2} = \left(\frac{\cos(S-B).\ \cos(S-C)}{\sin B.\ \sin C}\right)^{\frac{1}{2}}, \quad (2.)$$

$$\tan\frac{a}{2} = \left(\frac{-\cos S.\ \cos(S-A)}{\cos(S-B).\ \cos(S-C)}\right)^{\frac{1}{2}}, \quad (3.)$$

$$\sin a = \frac{2}{\sin B.\ \sin C}\{-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)\}^{\frac{1}{2}}; (4.)$$

the remarks in (art. 127. pt. I.) will show under what circumstances each of the above equations may be applied most advantageously. The results derived from the above equations are in degrees; let, therefore, the number of degrees corresponding to a be denoted by $A°$,

$$\text{then } A° = \frac{180°}{\pi}.\ a, \text{ rad} = 1, \text{ (art. 19. pt. I.)}$$

$$\therefore\ a = \frac{\pi}{180°}.\ A°;$$

$$\text{or } a' = \frac{\pi r}{180°}.\ A°;\ \text{rad} = r: .$$

thus the lengths of the arcs may be actually found to any given radius.

101. In (art. 76.) it was shown, that

$$\cos a = \frac{\cos A + \cos B.\ \cos C}{\sin B.\ \sin C};$$

hence, if X be assumed of such a magnitude, that

$$\cos X = \cos B.\ \cos C,$$

$$\therefore \cos a = \frac{\cos A + \cos X}{\sin B.\ \sin C},$$

$$= \frac{2\cos\left(\frac{A+X}{2}\right).\ \cos\left(\frac{A-X}{2}\right)}{\sin B.\ \sin C}; \text{ (art. 96. pt. I.)}$$

whence $\cos a$, and therefore a may be found.

The subsidiary single X is subtended by the hypothenuse of a right-angled triangle, of which the other two sides subtend the angles B and C, at the centre of the sphere; let the sides of this triangle be ξ, κ, λ;

$$\text{then } \cos \xi = \cos \kappa . \cos \lambda;$$

hence the same observations may be made concerning the magnitude of X, B, and C, that have been already made concerning the magnitude of θ, β, and γ.

102. To solve this case by *Napier's* rules, let P and Q be the angles into which the angle A is divided by the perpendicular let fall on a; then, by *Napier's* rules,

$$\cos B = \cos \epsilon . \sin Q,$$

$$\text{and } \cos C = \cos \epsilon . \sin P,$$

$$\therefore \frac{\cos B}{\cos C} = \frac{\sin Q}{\sin P};$$

$$\text{hence } \frac{\cos B - \cos C}{\cos B + \cos C} = \frac{\sin Q - \sin P}{\sin Q + \sin P},$$

$$\therefore \tan\left(\frac{B+C}{2}\right) . \tan\left(\frac{B-C}{2}\right) = \cot\left(\frac{Q+P}{2}\right) . \tan\left(\frac{Q-P}{2}\right),$$

$$= \cot \frac{A}{2} . \tan\left(\frac{Q-P}{2}\right),$$

$$\therefore \tan\left(\frac{Q-P}{2}\right) = \tan \frac{A}{2} . \tan\left(\frac{B+C}{2}\right) . \tan\left(\frac{B-C}{2}\right), \text{ and}$$

is therefore known;

$$\text{let } Q - P = D,$$

$$\text{and } Q + P = A,$$

$$\therefore Q = \frac{A+D}{2}, \text{ and } P = \frac{A-D}{2};$$

hence, by *Napier's* rules,

$$\cos \beta = \cot C . \cot\left(\frac{A-D}{2}\right),$$

$$\cos \gamma = \cot B . \cot\left(\frac{A+D}{2}\right).$$

$$\text{Also } \cos\epsilon = \frac{\cos B}{\sin\left(\frac{A+D}{2}\right)} = \frac{\cos C}{\sin\left(\frac{A-D}{2}\right)};$$

$$\text{and } \cos\frac{A+D}{2} = \cos Q.\ \sin B,$$

$$\therefore \cos Q = \frac{\cos\frac{A+D}{2}}{\sin B},$$

$$\text{similarly, } \cos P = \frac{\cos\left(\frac{A-D}{2}\right)}{\sin C};$$

thus the segments P and Q become known, and the oblique-angled triangle is completely solved by *Napier's* rules.

The remaining four cases may also be solved by the same rules, but it will not be necessary to pursue their application any further.

103. CASE III. *Given two sides and the included angle.*

Let C be the included angle,

α and β the given sides,

$$\text{then, } \tan\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}.\ \cot\frac{C}{2},$$

(art. 71.)

$$\tan\frac{A-B}{2} = \frac{\sin\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}.\ \cot\frac{C}{2};$$

From these two equations A and B are found by addition [and s]ubtraction.

104. If only one angle, A for example, be sought, it may be found from the equation,

$$\cot A = \cot \alpha . \sin \beta . \operatorname{cosec} C - \cos \beta . \cot C, \text{ (art. 84.)}$$

$$= \frac{\cot \alpha}{\sin C} . \left(\sin \beta - \frac{\cos C}{\cot \alpha} . \cos \beta\right),$$

let $\frac{\cos C}{\cot \alpha} = \tan \theta$,

where θ is the side of a right-angled triangle, α the hypothenuse, and C the included angle,

$$\text{then, } \cot A = \frac{\cot \alpha}{\sin C} (\sin \beta - \tan \theta . \cos \beta),$$

$$= \frac{\cot \alpha}{\sin C} . \quad \frac{\sin (\beta - \theta)}{\cos \theta}.$$

The remaining side γ may be found from the equation,

$$\sin \gamma = \frac{\sin C}{\sin A} . \sin \alpha, \text{ (art. 59.)}$$

Supposing the angle A already determined.

105. But it is preferable to find γ by one of the succeeding methods.

(1.) From (art. 61.),

$$\cos C = \frac{\cos \gamma - \cos \alpha . \cos \beta}{\sin \alpha . \sin \beta},$$

$$\therefore \cos \gamma = \cos \alpha . \cos \beta + \sin \alpha . \sin \beta . \cos C,$$

$$= \cos \alpha . \cos \beta + \sin \alpha . \sin \beta - \sin \alpha . \sin \beta . \operatorname{vers} C,$$

$$= \cos (\alpha - \beta) - \sin \alpha . \sin \beta . \operatorname{vers} C;$$

$$\text{hence, } 1 - \cos \gamma = \operatorname{vers} \gamma = \operatorname{vers} (\alpha - \beta) + \sin \alpha . \sin \beta . \operatorname{vers} C,$$

$$= \operatorname{vers} (\alpha - \beta) \left(1 + \frac{\sin \alpha . \sin \beta . \operatorname{vers} C}{\operatorname{vers} (\alpha - \beta)}\right);$$

$$\text{let } \frac{\sin \alpha . \sin \beta . \operatorname{vers} C}{\operatorname{vers} (\alpha - \beta)} = (\tan \theta)^2,$$

$$\therefore\ 2\left(\sin\frac{\gamma}{2}\right)^2 = \text{vers}\ \gamma = \text{vers}\ (\alpha - \beta)\ (\sec\theta)^2,\ \text{rad} = 1,$$

$$= \text{vers}\ (\alpha - \beta)\left(\frac{\sec\theta}{r}\right)^2,\ \text{rad} = r,$$

$$\text{or, } \sin\frac{\gamma}{2} = \sin\ \left(\frac{\alpha - \beta}{2}\right).\ \frac{\sec\theta}{r}.$$

(2.) $\text{Cos}\ \gamma = \cos\alpha.\ \cos\beta + \sin\alpha.\ \sin\beta.\ \cos C,$

$$= \cos\alpha.\ \cos\beta + \sin\alpha.\ \sin\beta.\left\{2\left(\cos\frac{C}{2}\right)^2 - 1\right\},$$

$$= \cos\ (\alpha + \beta) + 2\sin\alpha.\ \sin\beta.\ \left(\cos\frac{C}{2}\right)^2,$$

$$\therefore\ 1 - 2\left(\sin\frac{\gamma}{2}\right)^2 = 1 - 2\left\{\sin\left(\frac{\alpha+\beta}{2}\right)\right\}^2 + 2\sin\alpha.\ \sin\beta.\left(\cos\frac{C}{2}\right)^2,$$

$$\text{hence, } \left(\sin\frac{\gamma}{2}\right)^2 = \left\{\sin\left(\frac{\alpha+\beta}{2}\right)\right\}^2 - \sin\alpha.\ \sin\beta.\ \left(\cos\frac{C}{2}\right)^2,$$

$$\text{let } \sin\alpha.\ \sin\beta.\ \left(\cos\frac{C}{2}\right)^2 = (\sin\mu)^2,$$

$$\therefore\ \left(\sin\frac{\gamma}{2}\right)^2 = \left\{\sin\left(\frac{\alpha+\beta}{2}\right)\right\}^2 - (\sin\mu)^2,$$

$$= \left\{\sin\left(\frac{\alpha+\beta}{2}\right) + \sin\mu\right\}.\left\{\sin\left(\frac{\alpha+\beta}{2}\right) - \sin\mu\right\},$$

$$= 4\sin\frac{1}{2}\ \left(\frac{\alpha+\beta}{2} + \mu\right).\left\{\cos\frac{1}{2}\ \left(\frac{\alpha+\beta}{2} - \mu\right)\right\}.$$

(art. 95. pt. I.)

$$\times\cos\frac{1}{2}\ \left(\frac{\alpha+\beta}{2} + \mu\right).\ \left\{\sin\frac{1}{2}\ \left(\frac{\alpha+\beta}{2} - \mu\right)\right\},$$

$$= \sin\left(\frac{\alpha+\beta}{2} + \mu\right).\ \left\{\sin\left(\frac{\alpha+\beta}{2} - \mu\right)\right\};$$

$$\therefore\ \log.\ \sin\frac{\gamma}{2} = \frac{1}{2}\left\{\log.\ \sin\left(\frac{\alpha+\beta}{2} + \mu\right) + \log.\ \sin\left(\frac{\alpha+\beta}{2} - \mu\right)\right\}.$$

This is *Laplace's* method, but it does not give a result sufficiently accurate, when γ is nearly π, or $\frac{\gamma}{2} = \frac{\pi}{2}$ nearly.

(3.) If $\frac{\gamma}{2} = \frac{\pi}{2}$ nearly, it must be computed in the following manner: from the above,

$$\cos\gamma = \cos\alpha.\cos\beta + \sin\alpha.\sin\beta.\left\{1-2\left(\sin\frac{C}{2}\right)^2\right\},$$

$$= \cos(\alpha-\beta) - 2\sin\alpha.\sin\beta.\left(\sin\frac{C}{2}\right)^2,$$

$$\therefore 2\left(\cos\frac{\gamma}{2}\right)^2 - 1 = 2\left\{\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 - 1 - 2\sin\alpha.\sin\beta.\left(\sin\frac{C}{2}\right)^2,$$

$$\text{hence}\left(\cos\frac{\gamma}{2}\right)^2 = \left\{\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 - \sin\alpha.\sin\beta.\left(\sin\frac{C}{2}\right)^2,$$

$$\text{let } \sin\alpha.\sin\beta.\left(\sin\frac{C}{2}\right)^2 = (\cos\mu)^2,$$

$$\text{and}\left(\cos\frac{\gamma}{2}\right)^2 = \left\{\cos\left(\frac{\alpha-\beta}{2}\right)\right\}^2 - (\cos\mu)^2,$$

$$= \left\{\cos\left(\frac{\alpha-\beta}{2}\right) + \cos\mu\right\}.\left\{\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\mu\right\},$$

$$4\left\{\cos\frac{1}{2}\left(\frac{\alpha-\beta}{2}+\mu\right)\right\}.\left\{\cos\frac{1}{2}\left(\frac{\alpha-\beta}{2}-\mu\right)\right\},$$

$$\times\left\{\sin\frac{1}{2}\left(\frac{\alpha-\beta}{2}+\mu\right)\right\}.\left\{\sin\frac{1}{2}\left(\frac{\alpha-\beta}{2}-\mu\right)\right\},$$

(art. 96. pt. I.)

$$= \sin\left(\frac{\alpha-\beta}{2}+\mu\right).\left\{\sin\left(\frac{\alpha-\beta}{2}-\mu\right)\right\},$$

$$\therefore \log.\cos\frac{\gamma}{2} = \frac{1}{2}\left\{\log.\sin\left(\frac{\alpha-\beta}{2}+\mu\right) + \log.\sin\left(\frac{\alpha-\beta}{2}-\mu\right)\right\}.$$

(4.) Lastly, since

$$\cos\gamma = \cos\alpha.(\cos\beta + \tan\alpha.\sin\beta.\cos C),$$

if, as before, $\frac{\cos C}{\cot\alpha} = \tan\theta = \tan\alpha.\cos C$,

$$\cos\gamma = \cos\alpha.\frac{\cos(\beta-\theta)}{\cos\theta};$$

or, if $\dfrac{\cos C}{\cot a} = \cot \theta$,

$$\cos \gamma = \cos a. \frac{\sin (\beta + \theta)}{\sin \theta}.$$

106. CASE IV. *Given two angles and the included side.*

Let A and B be the two angles, and γ the included side; by (art. 81.),

$$\tan \left(\frac{a+\beta}{2}\right) = \frac{\cos \left(\frac{A-B}{2}\right)}{\cos \left(\frac{A+B}{2}\right)}. \tan \frac{\gamma}{2},$$

$$\tan \left(\frac{a-\beta}{2}\right) = \frac{\sin \left(\frac{A-B}{2}\right)}{\sin \left(\frac{A+B}{2}\right)}. \tan \frac{\gamma}{2};$$

hence, the sum and difference of the sides being found, the sides themselves are determined by addition and subtraction.

If only one side, a for example, be required, it may be found in the following manner: from (art. 84.),

$$\cot a = \cot A. \sin B. \operatorname{cosec} \gamma + \cos B. \cot \gamma,$$

$$= \frac{\cot A}{\sin \gamma} \left(\sin B + \frac{\cos \gamma}{\cot A}. \cos B\right),$$

Let $\dfrac{\cos \gamma}{\cot A} = \cot X$,

and $\cot a = \dfrac{\cot A}{\sin \gamma}. \left(\dfrac{\sin B. \sin X + \cos B. \cos X}{\cos X}\right)$,

$$= \frac{\cot A}{\sin \gamma}. \frac{\cos (B - X)}{\cos X}.$$

Observe, X and A are the angles of a right-angled triangle adjacent to the hypothenuse γ.

The remaining angle C may be found from the equation,

$$\sin C = \frac{\sin \gamma}{\sin a}. \sin A, \quad \text{(art. 59.)}$$

supposing the side a already determined.

107. But it is better to find C by one of the following methods, similar to those by which the side γ has been found in the preceding case.

(1.) From (art. 76.),

$$\cos \gamma = \frac{\cos C + \cos A. \cos B}{\sin A. \sin B},$$

$$\therefore \cos C = - \cos A. \cos B + \sin A. \sin B. \cos \gamma,$$

hence, $1 - \cos C = 1 + \cos A. \cos B - \sin A. \sin B. \cos \gamma,$

and $\text{vers } C = 1 + \cos (A + B) + \sin A. \sin B. \text{vers } \gamma,$

$$\therefore 2 \left(\sin \frac{C}{2}\right)^2 = 2 \left(\cos \frac{A + B}{2}\right)^2 + \sin A. \sin B. \text{vers } \gamma,$$

$$= 2 \left(\cos \frac{A + B}{2}\right)^2 . \left\{1 + \frac{\sin A. \sin B. \text{vers} \gamma}{2 \left(\cos \frac{A+B}{2}\right)^2}\right\},$$

let $\dfrac{\sin A. \sin B. \text{vers } \gamma}{2 \left(\cos \frac{A + B}{2}\right)^2} = (\tan \theta)^2,$

and $\sin \dfrac{C}{2} = \cos \left(\dfrac{A + B}{2}\right), \sec \theta, \text{rad} = 1,$

$$= \cos \left(\frac{A + B}{2}\right). \frac{\sec \theta}{r}, \text{rad} = r,$$

$$\therefore \log. \sin \frac{C}{2} = \log. \cos \left(\frac{A + B}{2}\right) + \log. \sec \theta - 10.$$

(2.) $\text{Cos } C = - \cos A. \cos B + \sin A. \sin B. \cos \gamma,$

$$= - \cos A. \cos B + \sin A. \sin B. \left\{2 \left(\cos \frac{\gamma}{2}\right)^2 - 1\right\},$$

$$= - \cos (A - B) + 2 \sin A. \sin B. \left(\cos \frac{\gamma}{2}\right)^2,$$

or, $-\cos C = \cos(A - B) - 2 \sin A. \sin B. \left(\cos \frac{\gamma}{2}\right)^2$,

$$\therefore 1 - \cos C = 2\left(\sin \frac{C}{2}\right)^2,$$

$$= 2\left(\cos \frac{A - B}{2}\right)^2 - 2 \sin A. \sin B. \left(\cos \frac{\gamma}{2}\right)^2,$$

$$\text{let } \sin A. \sin B. \left(\cos \frac{\gamma}{2}\right)^2 = (\cos M)^2,$$

$$\therefore \left(\sin \frac{C}{2}\right)^2 = \left\{\cos\left(\frac{A - B}{2}\right)\right\}^2 - (\cos M)^2,$$

$$= \sin\left(\frac{A - B}{2} + M\right). \sin\left(\frac{A - B}{2} - M\right),$$

$$\therefore \log. \sin \frac{C}{2} = \frac{1}{2}\left\{\log. \sin\left(\frac{A-B}{2} + M\right) + \log. \sin\left(\frac{A-B}{2} - M\right)\right\}.$$

If $\frac{C}{2} = 90°$ nearly, the following method must be used.

(3.) From the above,

$$\cos C = -\cos A. \cos B + \sin A. \sin B. \left\{1 - 2\left(\sin \frac{\gamma}{2}\right)^2\right\},$$

$$= -\cos(A+B) - 2 \sin A. \sin B. \left(\sin \frac{\gamma}{2}\right)^2,$$

$$= -1 + 2\left\{\sin\left(\frac{A + B}{2}\right)\right\}^2 - 2 \sin A. \sin B. \left(\sin \frac{\gamma}{2}\right)^2,$$

$$\text{hence, } 1 + \cos C = 2\left(\cos \frac{C}{2}\right)^2$$

$$= 2\left\{\sin\left(\frac{A + B}{2}\right)\right\}^2 - 2 \sin A. \sin B. \left(\sin \frac{\gamma}{2}\right)^2,$$

$$\text{or, } \left(\cos \frac{C}{2}\right)^2 = \left\{\sin\left(\frac{A + B}{2}\right)\right\}^2 - \sin A. \sin B. \left(\sin \frac{\gamma}{2}\right)^2,$$

$$\text{let } \sin A. \sin B. \left(\sin \frac{\gamma}{2}\right)^2 = (\sin M)^2,$$

$$\text{and } \left(\cos \frac{C}{2}\right)^2 = \left\{\sin\left(\frac{A + B}{2}\right)\right\}^2 - (\sin M)^2,$$

$$= \sin\left(\frac{A+B}{2} + M\right) . \sin\left(\frac{A+B}{2} - M\right),$$

$$\therefore \log. \cos\frac{C}{2} = \frac{1}{2}\left\{\log. \sin\left(\frac{A+B}{2} + M\right) + \log. \sin\left(\frac{A+B}{2} - M\right)\right\}.$$

(4.) Lastly, since

$$\cos C = -\cos A. \cos B + \sin A. \sin B. \cos \gamma,$$

$$= \sin A. \cos\gamma\left(\sin B - \frac{\cot A}{\cos\gamma}. \cos B\right),$$

let $\frac{\cot A}{\cos\gamma} = \tan X$,

$$\therefore \cos C = \frac{\sin A. \cos\gamma. \sin(B - X)}{\cos X}.$$

if $\frac{\cot A}{\cos\gamma} = -\cot X$,

$$\cos C = \frac{\sin A. \cos\gamma. \cos(B - X)}{\sin X}.$$

108. Case V. *Given two sides and an angle opposite to one of them.*

Let α and β be the given sides, and A the given angle,

$$\text{then } \frac{\sin B}{\sin A} = \frac{\sin\beta}{\sin\alpha}, \text{ (art. 59.)}$$

$$\therefore \sin B = \frac{\sin\beta}{\sin\alpha}. \sin A.$$

Because $\sin B = \sin(180° - B)$, the solution is ambiguous under the same circumstances, as in the corresponding case of plane triangles; but since it has been shown in (art. 75.) that $\frac{A+B}{2}$ is greater or less than 90°, according as $\frac{\alpha+\beta}{2}$ is greater or less than $\frac{\pi}{2}$, it follows that such a value must be given to B as will satisfy this condition; thus, if

$\frac{\alpha+\beta}{2}$ be greater than $\frac{\pi}{2}$,

and A $= 59°$, B $= 32°$, or $148°$,

then, since $\frac{59° + 32°}{2}$ is less than $90°$,

and $\frac{\alpha+\beta}{2}$ greater than $\frac{\pi}{2}$, by the supposition,

$\therefore$ B must $= 148°$ instead of $32°$.

Further, it can be shown, that of the two values of the angle opposite to β, that only can solve the problem which is greater or less than $90°$, according as β is greater or less than $\frac{\pi}{2}$.

By (art. 72.)

$$\cot \frac{C}{2} = \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \cdot \tan\left(\frac{A+B}{2}\right),$$

which determines the remaining angle C, the angle B being first found.

By (art. 82.)

$$\tan \frac{\gamma}{2} = \frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)} \cdot \tan\left(\frac{\alpha+\beta}{2}\right);$$

hence, the remaining side is also found; or it may be obtained from

$$\sin \gamma = \frac{\sin C}{\sin A} \cdot \sin \alpha = \frac{\sin C}{\sin B} \cdot \sin \beta.$$

109. But C and γ are better determined as follows:

$$\cot A . \sin C = \cot \alpha . \sin \beta - \cos \beta . \cos C, \text{ (art. 84.)}$$

$$\text{let } \tan X = \frac{\cos \beta}{\cot A}, \text{ or } \cos A = \frac{\cos \beta}{\tan X},$$

$$\therefore \frac{\cos\beta.\ \sin C}{\tan X} = \cot\alpha.\ \sin\beta - \cos\beta.\ \cos C,$$

or, $\cos\beta.\ \sin C.\ \cos X = \cot\alpha.\ \sin\beta.\ \sin X - \cos\beta.\ \cos C.\ \sin X$,
hence, $\cos\beta\ (\sin C.\ \cos X + \cos C.\ \sin X) = \cot\alpha.\ \sin\beta.\ \sin X$,

$$\text{or, } \cos\beta.\ \sin(C + X) = \cot\alpha.\ \sin\beta.\ \sin X,$$

$$\therefore \sin(C + X) = \cot\alpha.\ \tan\beta.\ \sin X,$$

from which equation $C + X$, and therefore C is found.

110. Also, since

$$\sin\beta.\ \sin\gamma.\ \cos A = \cos\alpha - \cos\beta.\cos\gamma, \text{ (art. 61.)}$$

$$\therefore \tan\beta.\ \cos A.\ \sin\gamma = \frac{\cos\alpha}{\cos\beta} - \cos\gamma,$$

$$\text{let } \tan\theta = \tan\beta.\ \cos A,$$

$$\therefore \sin\theta.\ \sin\gamma + \cos\theta.\cos\gamma = \frac{\cos\alpha.\ \cos\theta}{\cos\beta},$$

$$\text{or, } \cos(\gamma - \theta) = \frac{\cos\alpha.\ \cos\theta}{\cos\beta}:$$

since $\cos(\gamma - \theta) = \cos(\theta - \gamma)$, it is plain, that this result only gives the difference between γ and θ. From the assumed value of $\tan\theta$, it appears that β and θ are the hypothenuse and side of a right-angled triangle, and that A is the included angle. Hence, this triangle is formed by the side β and the perpendicular let fall from the angle C on γ; consequently, θ is the segment of the base between the foot of the perpendicular $(= \epsilon)$ and the angle A; and $\gamma - \theta$, when A and B are each less than a right angle, or $\theta - \gamma$, when one of the angles A and B is less, and the other greater than a right angle, is the other segment of the base between B and ϵ; since, in this case,

$$\frac{\cos(\gamma \backsim \theta)}{\cos\theta} = \frac{\cos\alpha}{\cos\beta}, \text{ (art. 98.)}$$

$$\text{or, } \cos(\gamma \backsim \theta) = \frac{\cos\alpha.\ \cos\theta}{\cos\beta};$$

which is the result just obtained. Hence, if the perpendicular fall *within* the base, the result obtained from

$$\frac{\cos a.\ \cos\theta}{\cos\beta} = \delta \text{ (suppose)} = \gamma - \theta,$$

$$\therefore \gamma = \theta + \delta;$$

if the perpendicular fall *without* the base,

$$\delta = \theta - \gamma,$$

$$\therefore \gamma = \theta - \delta.$$

Thus the above ambiguity is removed.

111. CASE VI. *Given two angles and the side opposite to one of them.*

Let A and B be the two angles,

and a the side opposite to A,

$$\text{then, } \sin\beta = \frac{\sin B}{\sin A}.\ \sin a;$$

the same considerations which were made use of in the last case to determine whether B or 180° — B ought to be taken, must be here applied to determine whether β or $\pi - \beta$ is the arc required.

As before, β being found,

$$\tan\frac{\gamma}{2} = \frac{\cos\left(\frac{A+B}{2}\right)}{\cos\left(\frac{A-B}{2}\right)}.\ \tan\left(\frac{a+\beta}{2}\right), \text{ (art. 82.)}$$

$$\text{and } \cot\frac{C}{2} = \frac{\cos\left(\frac{a+\beta}{2}\right)}{\cos\left(\frac{a-\beta}{2}\right)}.\ \tan\left(\frac{A+B}{2}\right).$$

Or, having found β, as in the last case,

$$\sin(C + X) = \cot a.\ \tan\beta.\ \sin X.$$

$$\text{and } \cos(\gamma \backsim \theta) = \frac{\cos a.\ \cos\theta}{\cos\beta}.\quad (a.)$$

112. The angle C may also be obtained thus,

$$\cot A = \cot \alpha.\ \sin\beta.\ \text{cosec } C - \cos\beta.\ \cot C,$$

$$\text{hence } \frac{\sin C.\ \cot A}{\cos \beta} = \cot\alpha.\ \tan\beta - \cos C\,;$$

$$\text{let } \frac{\cot A}{\cos \beta} = \tan X,$$

where X is the angle between the hypothenuse β and the perpendicular from the angle C on γ; hence,

$$\sin C.\ \sin X = \cot\alpha.\ \tan\beta.\ \cos X - \cos C.\ \cos X,$$

$$\therefore \cos(C \backsim X) = \cot\alpha.\ \tan\beta.\ \cos X. \quad (b.)$$

If the perpendicular fall *within* the base,

$$C - X = D, \text{suppose},$$

$$\text{and } \therefore C = X + D\,;$$

but if the perpendicular fall *without* the base,

$$X - C = D,$$

$$\text{and } \therefore C = X - D.$$

113. By applying the polar triangle to

$$\cos(C \backsim X) = \cot\alpha.\ \tan\beta.\ \cos X\,;$$

$$\text{and to } \cos(\gamma \backsim \theta) = \frac{\cos\alpha}{\cos\beta}.\ \cos\theta\,;$$

there results,

$$\sin(\gamma - \theta) = \cot A.\ \tan B.\ \sin\theta, \quad (1.)$$

$$\sin(C - X) = \frac{\cos A}{\cos B}.\ \sin X, \quad (2.)$$

$$\text{where } \cot\theta = \frac{\cot\alpha}{\cos B},$$

$$\text{and } \cot X = \tan B.\ \cos\alpha.$$

The equations (1.) and (2.) give the values of γ and C in terms of the *given* parts of the triangle.

SECTION V.

ON THE REMARKABLE REPRESENTATIONS OF THE SPHERICAL EXCESS, AND OTHER INTERESTING PROBLEMS.

114. PROP.

$$Sin\ \frac{E}{2} = \frac{sin\ \frac{\alpha}{2} .\ sin\ \frac{\beta}{2} .\ sin\ C}{cos\ \frac{\gamma}{2}}.$$

For, from (art. 56.),

$$E = (A + B + C - 180^\circ),$$

$$\therefore\ \frac{E}{2} = \left(\frac{A + B + C}{2} - 90^\circ\right),$$

$$\text{hence}\ \sin \frac{E}{2} = -\cos\left(\frac{E}{2} + 90^\circ\right),$$

$$= -\cos\left(\frac{A + B + C}{2}\right),$$

$$= -\cos\left(\frac{A + B}{2}\right).\ \cos\frac{C}{2} + \sin\left(\frac{A + B}{2}\right).\ \sin\frac{C}{2},$$

$$= -\left\{\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\frac{\gamma}{2}} - \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}}\right\}.\frac{\sin C}{2}, \text{(art. 70.)}$$

$$= \frac{\sin\frac{\alpha}{2}.\ \sin\frac{\beta}{2}.\ \sin C}{\cos\frac{\gamma}{2}}.$$

By substituting the value of sin C, in terms of the sides of the triangle,

$$\sin\frac{E}{2} = \frac{1}{2\cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \cos\frac{\gamma}{2}.}\{\sin\sigma.\ \sin(\sigma-\alpha).\ \sin(\sigma-\beta).\ \sin(\sigma-\gamma)\}^{\frac{1}{2}},$$

which is *Cagnole's* Theorem.

115. PROP.

$$Cos\,\frac{E}{2} = \frac{sin\,\frac{\alpha}{2}.\ sin\,\frac{\beta}{2}}{cos\,\frac{\gamma}{2}}.\ \left(cot\,\frac{\alpha}{2}.\ cot\,\frac{\beta}{2} + cos\,C\right).$$

For, $\cos\frac{E}{2} = \sin\left(\frac{E}{2} + 90^\circ\right)$,

$$= \sin\left(\frac{A+B+C}{2}\right),$$

$$= \sin\left(\frac{A+B}{2}\right).\ \cos\frac{C}{2} + \cos\left(\frac{A+B}{2}\right).\ \sin\frac{C}{2},$$

$$= \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\frac{\gamma}{2}}.\ \left(\cos\frac{C}{2}\right)^2 + \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\frac{\gamma}{2}}.\ \left(\sin\frac{C}{2}\right)^2,$$

$$= \frac{\cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \left(\cos\frac{C}{2}\right)^2}{\cos\frac{\gamma}{2}} + \frac{\sin\frac{\alpha}{2}.\ \sin\frac{\beta}{2}.\ \left(\cos\frac{C}{2}\right)^2}{\cos\frac{\gamma}{2}}$$

$$+ \frac{\cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \left(\sin\frac{C}{2}\right)^2}{\cos\frac{\gamma}{2}} - \frac{\sin\frac{\alpha}{2}.\ \sin\frac{\beta}{2}.\ \left(\sin\frac{C}{2}\right)^2}{\cos\frac{\gamma}{2}},$$

$$= \frac{\cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}}{\cos\frac{\gamma}{2}} + \frac{\sin\frac{\alpha}{2}.\ \sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}}.\ \cos C,$$

$$= \frac{\sin\frac{\alpha}{2}.\ \sin\frac{C}{2}}{\cos\frac{\gamma}{2}}.\ \left(\cot\frac{\alpha}{2}.\ \cot\frac{\beta}{2} + \cos C\right);$$

$$\text{hence}\quad \frac{\cos\frac{E}{2}}{\sin\frac{E}{2}} = \cot\frac{E}{2} = \frac{\cot\frac{\alpha}{2}.\ \cot\frac{\beta}{2} + \cos C.}{\sin C}.$$

116. This expression may also be derived in the following manner:

$$\cot\frac{E}{2} = -\tan\left(\frac{A+B+C}{2}\right),$$

$$= -\frac{\tan\left(\frac{A+B}{2}\right) + \tan\frac{C}{2}}{1 - \tan\left(\frac{A+B}{2}\right)\tan\frac{C}{2}},$$

$$\text{where}\quad \tan\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}.\ \cot\frac{C}{2};\ \text{(art. 71.)}$$

hence, by substituting for $\tan\left(\frac{A+B}{2}\right)$, and reducing, the required expression is found.

117. Again, since

$$\cot\frac{\alpha}{2}.\ \cot\frac{\beta}{2} = \left(\frac{1+\cos\alpha}{\sin\alpha}\right).\ \left(\frac{1+\cos\beta}{\sin\beta}\right),$$

$$= \frac{1+\cos\alpha+\cos\beta+\cos\alpha.\ \cos\beta}{\sin\alpha.\ \sin\beta},$$

$$\text{and}\ \cos C = \frac{\cos\gamma - \cos\alpha.\ \cos\beta}{\sin\alpha.\ \sin\beta},\ \text{(art. 61.)}$$

hence, by substitution and reduction,

$$\cot\frac{E}{2} = \frac{\cot\frac{\alpha}{2}.\cot\frac{\beta}{2} + \cos C}{\sin C},$$

$$= \frac{1 + \cos\alpha + \cos\beta + \cos\gamma}{2\,\{\sin\sigma.\ \sin(\sigma-\alpha).\ \sin(\sigma-\beta).\ \sin(\sigma-\gamma)\}^{\frac{1}{2}}},$$

which is *De Gua's* Theorem.

118. PROP.

$$Tan\frac{E}{4} = \left\{tan\frac{\sigma}{2}.\ tan\left(\frac{\sigma-\alpha}{2}\right).\ tan\left(\frac{\sigma-\beta}{2}\right).\ tan\left(\frac{\sigma-\gamma}{2}\right)\right\}^{\frac{1}{2}}.$$

Since $\sin\left(\frac{A+B}{2}\right) - \sin\left(90^\circ - \frac{C}{2}\right)$,

$$= 2\sin\left(\frac{A+B+C-180^\circ}{4}\right).\ \cos\left(\frac{A+B+180^\circ-C}{4}\right), \text{(I.;95.)}$$

$$\therefore\ 2\sin\frac{E}{2}.\ \cos\left(\frac{A+B+180^\circ-C}{4}\right) = \sin\left(\frac{A+B}{2}\right) - \cos\frac{C}{2},$$

$$= \frac{\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\frac{\gamma}{2}}{\cos\frac{\gamma}{2}}.\ \cos\frac{C}{2}, \text{ (art. 70.)}$$

$$= \frac{2\sin\left(\frac{\sigma-\alpha}{2}\right).\ \sin\left(\frac{\sigma-\beta}{2}\right)}{\cos\frac{\gamma}{2}}.\ \cos\frac{C}{2};\ \text{(I.; 96.)} \qquad (1.)$$

similarly, $2\cos\frac{E}{4}.\ \cos\left(\frac{A+B+180^\circ-C}{2}\right)$,

$$= \cos\left(\frac{A+B}{2}\right) + \cos\left(90^\circ - \frac{C}{2}\right),$$

$$= \cos\left(\frac{A+B}{2}\right) + \sin\frac{C}{2},$$

$$= \frac{2\cos\frac{\sigma}{2}.\ \cos\left(\frac{\sigma-\gamma}{2}\right)}{\cos\frac{\gamma}{2}}.\ \sin\frac{C}{2}, \qquad (2.)$$

hence, dividing (1.) by (2.),

$$\tan\frac{E}{4} = \frac{\sin\left(\frac{\sigma-\alpha}{2}\right).\ \sin\left(\frac{\sigma-\beta}{2}\right)}{\cos\frac{\sigma}{2}.\ \cos\left(\frac{\sigma-\gamma}{2}\right)}.\ \cot\frac{C}{2},$$

$$= \frac{\sin\left(\frac{\sigma-\alpha}{2}\right).\ \sin\left(\frac{\sigma-\beta}{2}\right)}{\cos\frac{\sigma}{2}.\ \cos\left(\frac{\sigma-\gamma}{2}\right)}.\ \left(\frac{\sin\sigma.\ \sin(\sigma-\gamma)}{\sin(\sigma-\alpha).\sin(\sigma-\beta)}\right)^{\frac{1}{2}},$$

$$= \frac{\sin\left(\frac{\sigma-\alpha}{2}\right).\ \sin\left(\frac{\sigma-\beta}{2}\right)}{\cos\frac{\sigma}{2}.\ \cos\left(\frac{\sigma-\gamma}{2}\right)}\times$$

$$\left\{\frac{\sin\frac{\sigma}{2}.\ \cos\frac{\sigma}{2}.\ \sin\left(\frac{\sigma-\gamma}{2}\right).\ \cos\left(\frac{\sigma-\gamma}{2}\right)}{\sin\left(\frac{\sigma-\alpha}{2}\right).\ \cos\left(\frac{\sigma-\alpha}{2}\right).\ \sin\left(\frac{\sigma-\beta}{2}\right).\ \cos\left(\frac{\sigma-\beta}{2}\right)}\right\}^{\frac{1}{2}},$$

$$= \left\{\tan\frac{\sigma}{2}.\ \tan\left(\frac{\sigma-\alpha}{2}\right).\ \tan\left(\frac{\sigma-\beta}{2}\right).\ \tan\left(\frac{\sigma-\gamma}{2}\right)\right\}^{\frac{1}{2}};$$

which is known by the name of *Lhuillier's* Theorem.

119. It is shown in the same manner, that

$$\tan\left(\frac{A+B-C+180^\circ}{4}\right) = \frac{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\frac{\gamma}{2}}{\cos\left(\frac{\alpha+\beta}{2}\right)+\cos\frac{\gamma}{2}}.\ \cot\frac{C}{2},$$

$$= \left\{\frac{\tan\frac{\sigma}{2}.\ \tan\left(\frac{\sigma-\gamma}{2}\right)}{\tan\left(\frac{\sigma-\alpha}{2}\right).\ \tan\left(\frac{\sigma-\beta}{2}\right)}\right\}^{\frac{1}{2}}.$$

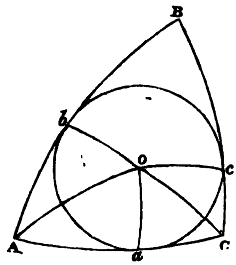
In (art. 110.) ε is the perpendicular from the angle C upon the opposite side, therefore $\sin\varepsilon = \sin\beta.\ \sin A$, by *Napier's* rules,

$$= \frac{2}{\sin \gamma} \{\sin \sigma . \sin (\sigma - \alpha) . \sin (\sigma - \beta) . \sin (\sigma - \gamma)\}^{\frac{1}{2}},$$

$$= \frac{2}{\sin C} \{-\cos S . \cos (S - A) . \cos (S - B) . \cos (S - C)\}^{\frac{1}{2}}.$$

120. **Prob.** *To inscribe a small circle in a given spherical triangle.*

Let ABC be the triangle, O the centre of the circle *bca*, inscribed in the triangle in the same manner as in the case of a plane triangle (Euc. IV. 4.)



O*b*, O*c*, O*a*, the perpendiculars on the sides AB, BC, AC; join OA, OC; then,

$$Ab + Bc + Ca = \frac{\alpha + \beta + \gamma}{2} = \sigma,$$

$$\therefore Ab = \sigma - (Bc + Ca) = \sigma - (Bc + Cc)$$

$$= \sigma - \alpha,$$

similarly, $Bc = \sigma - \beta$, and $Ca = \sigma - \gamma$;

let $Ob = Oc = Oa = r$; then, by *Napier's* rules,

$$\tan r = \tan \frac{A}{2} . \sin bA = \tan \frac{A}{2} . \sin (\sigma - \alpha);$$

$$\therefore \tan r = \sin (\sigma - \alpha) . \left\{\frac{\sin (\sigma - \beta) . \sin (\sigma - \gamma)}{\sin \sigma . \sin (\sigma - \alpha)}\right\}^{\frac{1}{2}}, (68.)$$

$$= \left\{\frac{\sin S . \sin (\sigma - \alpha) . \sin (\sigma - \beta) . \sin (\sigma - \gamma)}{\sin \sigma}\right\}^{\frac{1}{2}}:$$

hence, the segments of the sides, and the perpendicular dis-

tance of the pole from them, being found, the position of the pole is determined.

121. Let the sides BA, BC of the triangle ABC be produced through A and C, and let them meet in B_1: then, if $\sigma_1 = \frac{\alpha_1 + \beta_1 + \gamma_1}{2} = \frac{1}{2}$ sum of the sides of the triangle, so formed,

$$\sigma_1 = \pi - (\sigma - \beta),$$
$$\sigma_1 - \alpha_1 = \sigma - \gamma,$$
$$\sigma_1 - \beta_1 = \pi - \sigma,$$
$$\sigma_1 - \gamma_1 = \sigma - \alpha;$$

let r_1 = rad of inscribed circle,

$$\therefore \tan r_1 = \left\{\frac{\sin \sigma_1 . \sin (\sigma_1 - \alpha_1) . \sin (\sigma_1 - \beta_1) . \sin (\sigma_1 - \gamma_1)}{\sin S_1}\right\}^{\frac{1}{2}},$$
$$= \left\{\frac{\sin \sigma . \sin (\sigma - \alpha) . \sin (\sigma - \beta) . \sin (\sigma - \gamma)}{\sin (\sigma - \beta)}\right\}^{\frac{1}{2}};$$
$$= \frac{n}{\sin (\sigma - \beta)}, \text{ suppose.}$$

Similarly, if r_2 and r_3 be the radii of circles inscribed in the other two triangles, formed in the same way,

$$\tan r_2 = \frac{n}{\sin (\sigma - \alpha)},$$
$$\tan r_3 = \frac{n}{\sin (\sigma - \gamma)};$$

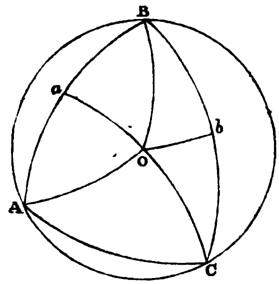
hence, $\tan r . \tan r_1 . \tan r_2 . \tan r_3$

$$= \frac{n}{\sin \sigma . \sin (\sigma - \alpha) . \sin (\sigma - \beta) . \sin (\sigma - \gamma)},$$
$$= \frac{n^4}{n^2} = n^2.$$

122. Prob. *To describe a small circle about a given spherical triangle.*

Let ABC be the circle described about the triangle in the

same manner as a circle is described about a plane triangle in (Euc. IV. 5.)



Let O be the pole of the circle; a and b the bisections of the sides AB, BC; join Oa, Ob, which are at right angles to AB, BC; join also OA, OB, OC; then,

$$2\,\text{OAB} + 2\,\text{OBC} + 2\,\text{OCA} = \text{A} + \text{B} + \text{C},$$

$$\therefore \text{OAB} + \text{OBC} + \text{OCA} = \frac{\text{A} + \text{B} + \text{C}}{2} = \text{S},$$

$$\begin{aligned} \text{hence, OAB} &= \text{S} - (\text{OBC} + \text{OCA}), \\ &= \text{S} - (\text{OCB} + \text{OCA}), \\ &= \text{S} - \text{C}; \end{aligned}$$

let R = radius of the circle,

then, by *Napier's* rules,

$$\cos \text{OAB} = \cos(\text{S} - \text{C}) = \tan\frac{\gamma}{2}.\cot \text{R},$$

$$\therefore \cot \text{R} = \cos(\text{S} - \text{C}).\cot\frac{\gamma}{2}.$$

Now, $\cos(\text{S} - \text{C}) = \cos\left(\frac{\text{A} + \text{B} - \text{C}}{2}\right)$,

$$= \cos\left(\frac{\text{A} + \text{B}}{2}\right).\cos\frac{\text{C}}{2} + \sin\left(\frac{\text{A} + \text{B}}{2}\right).\sin\frac{\text{C}}{2},$$

$$= \frac{1}{2}\left\{\cos\left(\frac{\alpha + \beta}{2}\right) + \cos\left(\frac{\alpha - \beta}{2}\right)\right\}.\frac{\sin \text{C}}{\cos\frac{\gamma}{2}}, \quad \text{(art. 70.)},$$

$$= \frac{\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}}{\cos\frac{\gamma}{2}}.\sin C, \quad \text{(art. 96. pt. I.)}$$

$$= \frac{2\cos\frac{\alpha}{2}.\cos\frac{\beta}{2}}{\cos\frac{\gamma}{2}.\sin\alpha.\sin\beta}\{\sin\sigma.\sin(\sigma-\alpha).\sin(\sigma-\beta).\sin(\sigma-\gamma)\}^{\frac{1}{2}},$$

$$= \frac{n}{2\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}.\cos\frac{\gamma}{2}},$$

$$\text{hence,} \cot R = \frac{n}{2\sin\frac{\alpha}{2}.\sin\frac{\beta}{2}.\sin\frac{\gamma}{2}}.$$

Or, since by (art. 80.),

$$\cot\frac{\gamma}{2} = \left(\frac{\cos(S-A).(\cos S-B)}{-\cos S.\cos(S-C)}\right)^{\frac{1}{2}}, \quad \text{(art. 80.)}$$

$$\therefore \cot R = \left(\frac{-\cos S.\cos(S-A).\cos(S-B).\cos(S-C)}{\cos S}\right)^{\frac{1}{2}}$$

$$= N, \text{ suppose.}$$

123. Let R_1, R_2, R_3, be the radii of circles described about the triangles formed by producing each pair of sides until they meet; then, as in (art. 121.),

$$\cot R_1 = \frac{N}{\cos(S-B)},$$

$$\cot R_2 = \frac{N}{\cos(S-A)},$$

$$\cot R_3 = \frac{N}{\cos(S-C)};$$

hence, $\cot R.\cot R_1.\cot R_2.\cot R_3 = N^2$.

124. PROB. *To find the locus of the vertex of a triangle of given area, with a given base.*

Let the sides AB, AC, be produced to meet in A_1, suppose

Let R_1 be the radius of the circle described about this triangle, and a the given base of the triangle ABC;

$$\text{then, } \cot R_1 = \cos(S_1 - A_1).\cot\frac{a_1}{2},$$

$$\text{where } S_1 = \frac{A_1+B_1+C_1}{2},$$

$$= \frac{A+180^\circ-B+180^\circ-C}{2}, \text{ for } A_1 = A,$$

$$= 180^\circ - \frac{B+C-A}{2},$$

$$\therefore S_1 - A_1 = 180^\circ - S,$$

$$\text{and } a_1 = a,$$

$$\therefore \cot R_1 = \cos(180^\circ - S).\cot\frac{a}{2},$$

$$= -\cos S.\cot\frac{a}{2};$$

which is a given quantity, since the area of the triangle being given, S is also given; therefore, the circle is given in magnitude; and, because it always passes through the extremities of the given base a, its position is given, and the vertex of the triangle BCA_1 is therefore in the circumference of this circle. Now, since the intersection of two great circles is a diameter, the line which joins the points A_1 and A is a diameter of the sphere; hence, since the extremity A_1 of this diameter traces out a small circle on the surface of the sphere, the point A likewise traces out another small circle equal and parallel to the former. The *locus*, therefore, of A, the vertex of the triangle ABC, is a *small circle* on the surface of the sphere.

125. Prob. *The vertical angle being given in magnitude and position, and also the perimeter, it is required to find the curve to which the base of a spherical triangle is always a tangent.*

In (art. 120.), let the sides AB, AC, which contain the given angle, be produced to meet in A_1, and let r_1 be the radius of the circle inscribed in this triangle;

$$\text{then, } \tan \gamma_1 = \tan \frac{A_1}{2} \cdot \sin (\sigma_1 - \alpha_1),$$

$$\text{where } A_1 = A, \text{ and } \alpha_1 = \alpha,$$

$$\text{also, } \sigma_1 = \frac{\pi - \beta + \pi - \gamma + \alpha}{2},$$

$$= \pi - \frac{\beta + \gamma - \alpha}{2},$$

$$\therefore \sigma_1 - \alpha_1 = \pi - \left(\frac{\beta + \gamma - \alpha}{2} + \alpha\right),$$

$$= \pi - \sigma,$$

$$\therefore \tan r_1 = \tan \frac{A}{2} \cdot \sin \sigma,$$

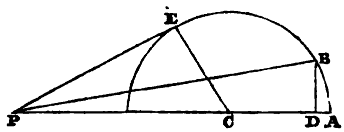
a constant quantity: the curve, therefore, to which the base of the triangle is always a tangent, is a circle, whose rad $= \tan \frac{A}{2} \cdot \sin \sigma$, and which touches the sides AB, and AC produced, and meeting at an angle equal to A.

126. Prob. *Given two sides and the included angle to find when the area is a maximum.*

From (art. 115.),

$$\cot \frac{E}{2} = \frac{\cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2} + \cos C}{\sin C}.$$

The following construction will show when this expression is a *maximum*.



Let ABE be a semicircle, C its centre; let the arc AB subtend at C an angle equal to C: draw BD perpendicular to CA, then

$$BD = \sin C, \text{ and } CD = \cos C.$$

Also, take $CP = \cot \frac{\alpha}{2} . \cot \frac{\beta}{2}$, join PB;

$$\text{then, } \frac{PD}{BD} = \frac{\cot \frac{\alpha}{2} . \cot \frac{\beta}{2} + \cos C}{\sin C},$$

$$= \cot \frac{E}{2} = \cot BPD,$$

hence, E is a maximum when the angle BPD is a maximum; that is, when BP is a tangent to the semicircle at E: join CE.

Hence, the angle $ACE - 90° =$ angle EPC,

$$\text{or, } C - 90° = \frac{E}{2} = \frac{A + B + C}{2} - 90°,$$

$$\text{consequently, } C = A + B;$$

that is, *the area is a maximum when the angle included by the given sides is equal to the sum of the two remaining angles.*

127. Prop. *If* C *be the spherical angle included between the arcs* α, β; *and* C′ *the angle between the chords* a, b; *then,*

$$\cos C' = \cos \frac{\alpha}{2} . \cos \frac{\beta}{2} . \cos C + \sin \frac{\alpha}{2} . \sin \frac{\beta}{2}.$$

For, $\cos\gamma = \sin\alpha.\ \sin\beta.\ \cos C + \cos\alpha.\ \cos\beta$, (art. 61.)

$$\therefore\ 1 - 2\left(\sin\frac{\gamma}{2}\right)^2 = 4\sin\frac{\alpha}{2}.\ \cos\frac{\alpha}{2}.\ \sin\frac{\beta}{2}.\ \cos\frac{\beta}{2}.\ \cos C$$

$$+\left\{1 - 2\left(\sin\frac{\alpha}{2}\right)^2\right\}.\ \left\{1 - 2\left(\sin\frac{\beta}{2}\right)\right\}^2,$$

or, because $C\,b\,\alpha\,\gamma = C = 2\sin\frac{\gamma}{2}$,

&c. = &c.

$$\therefore 1 - \frac{c^2}{2} = ab.\ \cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \cos C + \left(1 - \frac{a^2}{2}\right).\ \left(1 - \frac{b^2}{2}\right),$$

$$= ab.\ \cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \cos C + 1 - \frac{a^2}{2} - \frac{b^2}{2} + \frac{a^2b^2}{4},$$

$$\therefore \frac{a^2 + b^2 - c^2}{2ab} = \cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \cos C + \frac{ab}{4},$$

hence, $\cos C' = \cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}.\ \cos C + \sin\frac{\alpha}{2}.\ \sin\frac{\beta}{2}.$

128. Cor. 1. If $C = 0$,

$$\cos C' = \cos\left(\frac{\alpha - \beta}{2}\right):$$

if $C = 180°$,

$$\cos C' = -\cos\left(\frac{\alpha + \beta}{2}\right);$$

in both these cases the arcs α and β are in the same plane of a great circle; in the former case, C' is greater than C; and in the latter, C' is less than C. Also, if the spherical angle be equal to 90°, the angle between the chords is then less than 90°.

129. Cor. 2. $$\operatorname{Cos} C = \frac{\cos C' - \sin\frac{\alpha}{2}.\ \sin\frac{\beta}{2}}{\cos\frac{\alpha}{2}.\ \cos\frac{\beta}{2}},$$

$$= \frac{\cos C' - \dfrac{ab}{4}}{\sqrt{\left\{\left(1 - \dfrac{a^2}{4}\right).\left(1 - \dfrac{b^2}{4}\right)\right\}}},$$

$$= \frac{4 \cos C' - ab}{\sqrt{\{(4 - a^2).(4 - b^2)\}}};$$

which gives the value of cos C in terms of the chords of the two arcs, and the included angle.

If $a = b$,

$$\cos C = \frac{4 \cos C' - a^2}{4 - a^2}.$$

SECTION VI.

ON THE MEASURING OF SOLID ANGLES.

130. Prop. *If* S *be a solid angle, and* A *the spherical surface to radius unity subtending it; then,*

$$S = \frac{180^\circ}{\pi}. A.$$

A solid angle being the angular space made by the meeting of more than two plane angles, which are not in the same plane, in one point; if about this point as a centre, a spherical surface be described, it is plain that the solid angle will bear the same relation to the corresponding spherical surface, that the plane angle bears to the circular arc by which it is subtended: hence the magnitudes of solid angles may also be compared by determining their corresponding spherical surfaces to the same radius. Now, in (art. 57.), it was shown, that the area of a spherical polygon of n sides

$$= \frac{\pi}{180^\circ} \{\text{angles of polygon} - (n-2)\ 180^\circ\},$$

where the angles of the spherical polygon are the same as the angles at which the planes, that form the solid angle, are inclined to each other. And the greatest limit of a solid angle is obviously 360°, and its measure the surface of the hemisphere ($= 2\pi$), in the same manner as the greatest limit of a

plane angle is 180°, and its measure a semicircle, or π, to radius unity. But if a be the arc corresponding to a plane angle $= A$,

$$\frac{A}{180^\circ} = \frac{a}{\pi};$$

$$\therefore \frac{S}{360^\circ} = \frac{a}{2\pi},$$

$$\text{hence } S = \frac{180^\circ}{\pi}. a.$$

131. Cor. If $a = n a'$, or the spherical surface be divided into n equal portions, then

$$\frac{S}{n} = \frac{180^\circ}{\pi}. a'.$$

132. It only remains to apply this expression for the value of a solid angle to a few familiar instances.

(1.) *To find the solid angle of a cube.*

Since the planes which form the solid angle are at right angles to each other, the spherical surface ($= a$), which subtends the solid angle,

$$= \frac{\pi}{180^\circ} (3.\ 90^\circ - 2.\ 90^\circ),$$

$$= \frac{\pi}{2},$$

$$\therefore S = \frac{180^\circ}{\pi}. a = 90^\circ;$$

which is one fourth of the angular space about a point on one side of a plane. Hence, four cubes fully occupy the angular space about a point.

(2.) *To find the solid angle of a regular right prism with a triangular base.*

In this case,

$$a = \frac{\pi}{180^\circ}(2.90^\circ + 60^\circ - 2.90^\circ),$$

$$= \frac{\pi}{3},$$

$$\therefore S = 60^\circ,$$

hence, the solid angle of this prism

$$= \frac{2}{3} \text{ of solid angle of cube.}$$

(3.) *If the right regular prism have* n *sides, it is required to find the solid angle.*

Since each of the angles of the base of the prism

$$= \frac{n-2}{n}.\ 180^\circ, \text{ (Euc. I. 32. Cor. 1.)}$$

$$a = \frac{\pi}{180^\circ}.\left(2.90^\circ + \frac{n-2}{n}.\ 180^\circ - 2.90^\circ\right),$$

$$= \frac{\pi}{180^\circ}.\ \frac{n-2}{n}.\ 180^\circ,$$

$$\therefore S = \frac{n-2}{2n}.\ 360^\circ.$$

Hence, the sum of all the solid angles,

$$= n\,S = \frac{n-2}{2}.\ 360^\circ.$$

(4.) *To find the solid angle at the vertex of a regular pyramid of* n *sides.*

Let A = inclination of two contiguous sides,

then, $S = \frac{180^\circ}{\pi}.\ a = 360^\circ - n\,(180^\circ - A)$.

If the pyramid be not a regular one, and A, B, C, &c. be the angles of inclination,

$$S = 360^\circ - n.\ 180^\circ + A + B + C + \ldots.$$

(5.) *To find the solid angle at the vertex of a cone.*

Let B be the vertical angle of the cone, then the part of the diameter of the sphere to radius unity, described about the vertex of the cone, corresponding to the spherical segment within the cone,

$$= 1 - \cos\frac{B}{2};$$

$$\therefore\ a = 2\pi\left(1 - \cos\frac{B}{2}\right),\quad \text{Diff. Cal.}$$

$$\text{hence, } S = \frac{180^\circ}{\pi}.\ a,$$

$$= \frac{180^\circ}{\pi}.\ 2\pi\left(1 - \cos\frac{B}{2}\right),$$

$$= 360^\circ.\left(1 - \cos\frac{B}{2}\right).$$

If the cone be equilateral, $\frac{B}{2} = 30^\circ$,

$$\therefore\ S = 360^\circ.\left(\frac{2 - 3^{\frac{1}{2}}}{2}\right).$$

If it be right-angled, $\frac{B}{2} = 45^\circ$,

$$\therefore\ S_1 = 360^\circ.\left(\frac{2^{\frac{1}{2}} - 1}{2^{\frac{1}{2}}}\right),$$

$$\therefore\ \frac{S}{S_1} = \frac{2 - 3^{\frac{1}{2}}}{2 - 2^{\frac{1}{2}}}.$$

SECTION VII.

ON THE SMALL CORRESPONDING VARIATIONS OF THE PARTS OF A TRIANGLE.

133. To estimate the probable effect of error in observation; to reduce observations made in one situation to what they would be in a situation little distant; to take account of refraction, parallax, precession, &c., it is absolutely necessary to ascertain the effect which will be produced on one part of a triangle by the variation of another, all the rest remaining unaltered. In almost all cases expressions may be conveniently found by writing down two equations, one of which results from giving to the quantities contained in the other the variations which they are supposed to undergo, and then taking the difference. The advantage of this method consists in showing precisely the magnitude of the error made by any further simplification. The following examples will point out more clearly the meaning of what has been just observed. In this chapter, A, B, C are circular arcs to radius unity.

134. If the given equation be

$$\cos A = \frac{\cos \alpha - \cos \beta . \cos \gamma}{\sin \beta . \sin \gamma},$$

and ΔA, and Δa, be the contemporary variations of A and a; β and γ remaining unvaried; then

$$\cos(A+\Delta A) = \frac{\cos(a+\Delta a) - \cos\beta.\cos\gamma}{\sin\beta.\sin\gamma};$$

$\therefore$ by subtraction,

$$\cos(A+\Delta A) - \cos A = \frac{\cos(a+\Delta a) - \cos a}{\sin\beta.\sin\gamma};$$

but, by *Taylor's* Theorem,

$$\cos(A+\Delta A) - \cos A = -\sin A.\Delta A - \cos A.\frac{(\Delta A)^2}{1.2} + \ldots$$

$$\text{and} \cos(a+\Delta a) - \cos a = -\sin a.\Delta a - \cos a.\frac{(\Delta a)^2}{1.2} + \ldots$$

$$\text{hence, } \sin A.\Delta A + \cos A.\frac{(\Delta A)^2}{1.2} - \ldots = \frac{\sin a.\Delta a + \cos a.\frac{(\Delta a)^2}{1.2} -}{\sin\beta.\sin\gamma}.$$

In almost all cases, ΔA, and therefore Δa is so small that it is sufficient to take only the first terms of the series on each side of this equation; in which case,

$$\sin A.\Delta A = \frac{\sin a.\Delta a}{\sin\beta.\sin\gamma},$$

$$\text{and } \therefore \frac{\Delta A}{\Delta a} = \frac{\sin a}{\sin A.\sin\beta.\sin\gamma};$$

hence, if the magnitude either of ΔA, or Δa be given, the magnitude of the other variation is found.

If the philosophical problem require great accuracy, another and more correct approximate value of ΔA, or Δa, may be obtained by retaining the first two terms of the above series, and finding the value of ΔA in terms of Δa, or Δa in terms of ΔA, by the solution of a common quadratic equation.

135. Again, if a be the measured distance from the base of a building; θ the arc subtending the observed angle of

elevation, and x the required altitude; then

$$x = a \tan\theta;$$

let Δx and $\Delta\theta$ be the contemporary variations of x and θ, and

$$x + \Delta x = a \tan(\theta + \Delta\theta),$$

$$\therefore \quad \Delta x = a\,\{\tan(\theta + \Delta\theta) - \tan\theta\},$$

$$= a \sec\theta.\ \sec(\theta + \Delta\theta).\ \sin\Delta\theta, \text{ (art. 101. pt.I.)}$$

and $\Delta x = a\,(\sec\theta)^2\,\Delta\theta$; $\Delta\theta$ being, from the nature of the case, very small: thus the error of observation being given, the error in altitude is found.

136. **Prob.** *To find when this error is the least.*

By the supposition a is indeterminate, and

$$= x \cot\theta, \text{ by the preceding article,}$$

$$\therefore \quad \Delta x = x\,\frac{\cos\theta}{\sin\theta}.\ \frac{\Delta\theta}{(\cos\theta)^2},$$

$$= \frac{2\,x\,\Delta\theta}{\sin 2\,\theta};$$

which is least when $\sin 2\,\theta$ is the greatest; that is, when

$$2\theta = \frac{\pi}{2},$$

$$\text{or, } \theta = \frac{\pi}{4};$$

which shows that the error in altitude is least when the angle of elevation is 45°; or when the height of the building is equal to the distance of the observer from its base.

137. In the equation,

$$\frac{\sin A}{\sin C} = \frac{\sin\alpha}{\sin\gamma}; \quad \text{(art 59.)}$$

let $C = \frac{\pi}{2}$, and

$\sin a = \sin A . \sin \gamma$,

$\therefore \sin (a + \Delta a) = \sin A . \sin (\gamma + \Delta \gamma)$, A being given;

$\therefore \sin (a + \Delta a) - \sin a = \sin A \{(\sin \gamma + \Delta \gamma) - \sin \gamma)\}$,

hence, $\cos\left(a + \frac{\Delta a}{2}\right). \sin \frac{\Delta a}{2} = \sin A . \cos\left(\gamma + \frac{\Delta \gamma}{2}\right). \sin \frac{\Delta \gamma}{2}$,

or, if Δa and $\Delta \gamma$ be very small,

$$\Delta a = \frac{\sin A . \cos \gamma}{\cos a} . \Delta \gamma,$$

$$= \sin A . \cos \beta . \ \Delta \gamma; \quad \text{(art. 62.)}$$

$$\left.\begin{array}{l} \text{but } \sin A = \dfrac{\sin a}{\sin \gamma}, \\ \text{and } \cos \beta = \dfrac{\cos \gamma}{\cos a}, \end{array}\right\} \text{by } \textit{Napier's} \text{ rules,}$$

$$\therefore \ \Delta a = \frac{\tan a}{\tan \gamma} . \Delta \gamma,$$

if m and n be the number of seconds in Δa and $\Delta \gamma$, respectively,

$$m = n \frac{\tan a}{\tan \gamma}.$$

If $\gamma = \frac{\pi}{2}$, care must be taken in the reduction, and it must not be supposed, that, because $\sin a = \sin A$ in this case, $\delta a = 0$; for, from the above,

$$\cos\left(a + \frac{\Delta a}{2}\right). \sin \frac{\Delta a}{2} = - \sin A . \left(\sin \frac{\Delta \gamma}{2}\right)^2,$$

$$\text{hence, } \Delta a = - \frac{\sin A}{\cos a} . \frac{(\Delta \gamma)^2}{2},$$

$$= - \tfrac{1}{2} \tan a \, (\Delta \gamma)^2,$$

$$= - \tfrac{1}{2} \tan A \, (\Delta \gamma)^2;$$

$$\text{or, } m = - 0,000002424 \times n^2 \tan A;$$

where 0.000004848 is the arc of $1''$ corresponding to rad $= 1$.

138. **Prop.** *If* $\alpha = f\gamma$, *and* $\Delta\alpha$, $\Delta\gamma$, *be the corresponding variations of* α *and* γ; *where* α *and* γ *are both trigonometric functions, then*

$$\Delta\alpha = \frac{d\alpha}{d\gamma}.\ \Delta\gamma + \frac{d^2\alpha}{d\gamma^2}.\ \frac{(\Delta\gamma)^2}{1.\ 2} + \frac{d^3\alpha}{d\gamma^3}.\ \frac{(\Delta\gamma)^3}{1.2.3}\cdots$$

By the supposition,

$$\alpha + \Delta\alpha = f(\gamma + \Delta\gamma),$$

$$= f\gamma + \frac{d.f\gamma}{d\gamma}.\ \Delta\gamma + \frac{d^2 f\gamma}{d\gamma^2}.\ \frac{(\Delta\gamma)^2}{1.\ 2}\cdots$$

$$\text{or, } \alpha + \Delta\alpha = \alpha + \frac{d\alpha}{d\gamma}.\ \Delta\gamma + \frac{d^2\alpha}{d\gamma^2}.\ \frac{(\Delta\gamma)^2}{1.\ 2}\cdots$$

$$\therefore\ \Delta\alpha = \frac{d\alpha}{d\gamma}.\ \Delta\gamma + \frac{d^2\alpha}{d\gamma^2}.\ \frac{(\Delta\gamma)^2}{1.\ 2} + \frac{d^3\alpha}{d\gamma^3}.\ \frac{(\Delta\gamma)^3}{1.\ 2.\ 3}\cdots$$

If $\Delta\gamma$ be very small,

$$\Delta\alpha = \frac{d\alpha}{d\gamma}.\ \Delta\gamma, \text{ very nearly.}$$

If, however, $\frac{d\alpha}{d\gamma} = 0$,

$$\text{then, } \Delta\alpha = \frac{d^2\alpha}{d\gamma^2}.\ \frac{(\Delta\gamma)^2}{1.\ 2}, \text{ nearly.}$$

139. Thus, in (art. 135.), where $\sin\alpha = \sin A.\ \sin\gamma$,

$$\frac{d\alpha}{d\gamma} = \frac{\sin A.\ \cos\gamma}{\cos\alpha}, \quad (1.)$$

$$\therefore\ \Delta\alpha = \frac{\sin A.\ \cos\gamma}{\cos\alpha}.\ \Delta\gamma, \text{ nearly.}$$

Again, from (1.),

$$\frac{d^2\alpha}{d\gamma^2}.\cos\alpha - \sin\alpha.\ \frac{(d\alpha)^2}{d\gamma} = -\sin A.\ \sin\gamma,$$

$$\therefore\ \frac{d^2\alpha}{d\gamma^2}.\ \cos\alpha = \frac{\sin\alpha.\ (\sin A)^2.\ (\cos\gamma)^2}{(\cos\alpha)^2} - \sin A.\ \sin\gamma,$$

let $\gamma = \frac{\pi}{2}$, then $\sin\alpha = \sin A$, and $\cos\alpha = \cos A$,

$$\therefore \frac{d^2 a}{d\gamma^2} = -\tan A;$$

hence, $\Delta a = -\tan A \frac{(\Delta \gamma)^2}{1.2}$, nearly, as before.

This method is convenient when the first differential coefficient does not vanish, and when the neglect of the other terms will certainly introduce no error; but when a particular value makes the first differential coefficient vanish, or when it is necessary to examine the terms after the first, the method illustrated by the preceding articles is generally to be preferred.

140. By referring to articles 132 and 135, it appears from the examples there given, that

$$\frac{\Delta A}{\Delta a} = \frac{\sin a}{\sin A.\sin\beta.\sin\gamma}, \quad (1.)$$

$$\text{and } \frac{\Delta a}{\Delta \gamma} = \frac{\sin A.\ \cos\gamma}{\cos a}; \quad (2.)$$

by taking the limiting ratio of these variations, there results from (1.),

$$\frac{dA}{da} = \frac{\sin a}{\sin A.\sin\beta.\sin\gamma},$$

and from (2.), $\frac{da}{d\gamma} = \frac{\sin A.\cos\gamma}{\cos a}$;

hence, it follows, that the ratio of the errors is expressed by the ratio of the differentials of the variable parts in the triangle, and may therefore be found by simple differentiation, on the conditions specified at the close of the last article.

141. In a right-angled triangle there are three sides and two angles which are subject to variation; hence any one of these five parts remaining constant, the limiting ratios of the contemporary variations of the rest may be expressed. And

it is plain that in a right-angled triangle the five cases are included in three; which are,

(1.) When the hypothenuse is given.

(2.) When a side about the right angle is given.

(3.) When an angle adjacent to the hypothenuse is given.

Since the investigation of these cases is so simple, it will suffice to discuss only one of them.

142. PROB. *Let the hypothenuse of a right-angled triangle be given; it is required to find the limiting ratio of the contemporary variations of the other parts.*

By *Napier's* rules;

(1.) $\cot A = \tan B. \cos \gamma$, where $\cos \gamma$ is given,

$$\therefore \frac{dA}{dB} = \cos \gamma . \left(\frac{\sec B}{\operatorname{cosec} A}\right)^2; \qquad (1.)$$

(2.) $\sin \alpha = \sin A. \sin \gamma$,

$$\therefore \frac{dA}{d\alpha} = \frac{\cos \alpha}{\cos A. \sin \gamma},$$

$$= \frac{\cos \alpha. \sin A}{\sin \alpha. \cos A},$$

$$= \frac{\tan A}{\tan \alpha}; \qquad (2.)$$

(3.) $\cos A = \tan \beta. \cot \gamma$,

$$\therefore \frac{dA}{d\beta} = -\frac{(\sec \beta)^2. \cot \gamma}{\sin A},$$

$$= -\frac{(\sec \beta)^2. \cos A}{\sin A. \tan \beta};$$

$$= -\frac{2 \cot A}{\sin 2\beta}; \qquad (3.)$$

$$\text{similarly, } \frac{dB}{d\alpha} = -\frac{2 \cot \beta}{\sin 2\alpha}; \qquad (4.)$$

$$\frac{dB}{d\beta} = \frac{\tan B}{\tan \beta}; \qquad (5.)$$

(4.) lastly, $\cos\gamma = \cos a.\ \cos\beta$,

$\therefore\ \cos a = \cos\gamma.\ \sec\beta$,

hence $-\sin a\, d\,a = \cos\gamma.\ \sec\beta.\ \tan\beta\, d\,\beta$,

$$\therefore \frac{d\,a}{d\,\beta} = -\frac{\cos\gamma.\ \sec\beta.\ \tan\beta}{\sin a},$$

$$= -\frac{\cos a.\ \sin\beta}{\sin a.\ \cos\beta} = -\frac{\tan\beta}{\tan a}. \quad (6.)$$

143. In the two remaining cases, by a similar application of *Napier's* rules, and by differentiation, it is readily shown, a being given, that,

$$\frac{d\,A}{d\,B} = -\frac{\cot B}{\tan A}, \quad \frac{d\,A}{d\,\gamma} = -\frac{\tan A}{\tan\gamma},$$

$$\frac{d\,A}{d\,\beta} = -\frac{\cos\beta}{\tan a.\ (\operatorname{cosec} A)^2}, \quad \frac{d\,B}{d\,\beta} = \frac{\sin 2B}{\sin 2\beta},$$

$$\frac{d\,\gamma}{d\,\beta} = \frac{\tan\beta}{\tan\gamma}, \ \&c.$$

If β be given instead of a, similar equations may be shown to subsist. The negative sign shows that one of the variable quantities is decreasing.

144. If the angle A be given; then

$$\frac{d B}{d\,a} = \frac{\tan B}{\cot a}, \quad \frac{d\,B}{d\,\beta} = \frac{\tan\beta}{\tan B},$$

$$\frac{d B}{d\,\gamma} = \frac{\sin 2B}{2\cot\gamma}, \quad \frac{d\,a}{d\,\beta} = \frac{\sin 2a}{2\tan\beta},$$

$$\frac{d\,a}{d\,\gamma} = \frac{\tan a}{\tan\gamma}, \ \&c.$$

If the angle B be given instead of A, similar equations may be found.

145. In an oblique-angled triangle there are six parts, namely, the three sides and the three angles, which are subject to variation; hence, any two of these six parts remaining con-

stant, the limiting ratios of the contemporary variations of the remaining four parts may be found in six equations. It is manifest, that all the cases of variation in the parts of an oblique-angled triangle are contained in the following four:

(1.) When two sides are given.

(2.) When two angles are given.

(3.) When an angle and the adjacent side is given.

(4.) When an angle and the opposite side is given.

It will not be necessary to examine these four cases in detail, on account of the similarity of operation.

146. PROB. *Let the two given sides be α and β, it is required to find the limiting ratio of the contemporary variations of the other parts.*

(1.) Since $\frac{\sin A}{\sin B} = \frac{\sin \alpha}{\sin \beta}$,

$$\therefore \sin A = \frac{\sin \alpha}{\sin \beta} . \sin B,$$

$$\text{hence } \frac{dA}{dB} = \frac{\sin \alpha}{\sin \beta} . \frac{\cos B}{\cos A},$$

$$= \frac{\sin A}{\sin B} . \frac{\cos B}{\cos A},$$

$$= \frac{\tan A}{\tan B}; \qquad (1.)$$

(2.) $$\cos C = \frac{\cos \gamma - \cos \alpha . \cos \beta}{\sin \alpha . \sin \beta},$$

$$\therefore \frac{dC}{d\gamma} = \frac{\sin \gamma}{\sin \alpha . \sin \beta . \sin C},$$

$$= \frac{1}{\sin \beta . \sin A} = \frac{1}{\sin \alpha . \sin B}; \qquad (2.)$$

(3.) $$\cos A = \frac{\cos \alpha - \cos \beta . \cos \gamma}{\sin \beta . \sin \gamma},$$

hence, by differentiation and reduction,

$$\frac{dA}{d\gamma} = \frac{-\cos B.\ \sin \alpha}{\sin \beta.\ \sin \gamma.\ \sin A} = \frac{-\cos B}{\sin B.\ \sin \gamma} = \frac{-\cot B}{\sin \gamma}; \quad (3.)$$

(4.) since, from (2.),

$$\frac{d\gamma}{dC} = \sin \alpha.\ \sin B,$$

$$\therefore \frac{dA}{dC} = \frac{-\cos B.\sin \alpha.\ \sin B}{\sin B.\ \sin \gamma} = -\frac{\cos B.\ \sin A}{\sin C}. \quad (4.)$$

(5.) from (1.) and (4.)

$$\frac{dB}{dC} = -\frac{\cos A.\ \sin B}{\sin C}, \quad (5.)$$

(6.) since, $\cos B = \dfrac{\cos \beta - \cos \alpha.\ \cos \gamma}{\sin \alpha.\ \sin \gamma}$,

by the same operation as before,

$$\frac{dB}{d\gamma} = -\frac{\cot A}{\sin \gamma}. \quad (6.)$$

147. For the three remaining cases it will be sufficient to write down the equations, which express the small contemporary variations.

When two angles as A *and* B *are given; then,*

$$\frac{dC}{d\alpha} = \frac{\sin C}{\cot \beta}, \quad \frac{dC}{d\gamma} = \frac{\sin C}{\cot \alpha},$$

$$\frac{dC}{d\gamma} = \sin A.\ \sin \beta, \quad \frac{d\alpha}{d\beta} = \frac{\tan \alpha}{\tan \beta},$$

$$\frac{d\alpha}{d\gamma} = \frac{\cos \beta.\ \sin A}{\sin C}, \quad \frac{d\beta}{d\gamma} = \frac{\cos \alpha.\ \sin B}{\sin C}.$$

148. *When the angle* A *and the adjacent side* β *are given;*

$$\frac{dB}{dC} = -\cos \alpha, \quad \frac{dB}{d\alpha} = -\frac{\tan B}{\tan \alpha},$$

$$\frac{d\alpha}{d\gamma} = \cos B, \quad \frac{dC}{d\gamma} = \frac{\sin B}{\sin \alpha}.$$

&c. = &c.

149. *When the angle* A *and the opposite side* a *are given;*

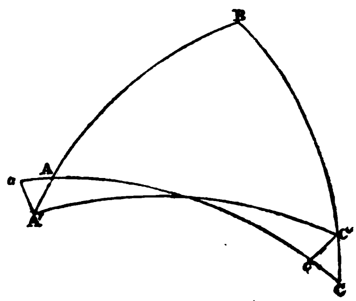
$$\frac{d\,\mathrm{B}}{d\,\mathrm{C}} = -\frac{\cos\beta}{\cos\gamma}, \quad \frac{d\,\mathrm{B}}{d\,\beta} = \frac{\tan \mathrm{B}}{\tan\beta},$$

$$\frac{d\,\beta}{d\,\gamma} = -\frac{\cos \mathrm{B}}{\cos \mathrm{C}}, \quad \frac{d\,\mathrm{B}}{d\,\gamma} = -\frac{\sin \mathrm{B}}{\tan\beta.\cos \mathrm{C}},$$

$$\&c. = \&c.$$

150. As soon as the student has become familiar with astronomical terms, he will be able to apply the principles of this section to a variety of problems, of great interest and utility, in the science of astronomy.

151. In the solution of problems, it is often advantageous to find the limiting ratio of the small corresponding variations *geometrically*, by considering the small triangles, the parts of which represent the contemporary variations, as rectilinear. Thus, if the triangle ABC be changed into the triangle



A′BC′, by the small variations of its sides BA, BC, whilst the angle B and the opposite side AC continue unchanged; and, if perpendiculars A′a, C′c be let fall from A′, C′ on AC produced towards A; because A′C′ = AC by the supposition, therefore Aa = Cc: and the triangles being regarded as rectilinear,

$$\mathrm{A}a = \mathrm{AA}' \cos \mathrm{BAC},$$

$$\text{and } Cc = -\ CC' \cos BCA,$$

$$\therefore \frac{CC'}{AA'} = -\ \frac{\cos BAC}{\cos BCA},$$

$$\text{or } \frac{d\,a}{d\,\gamma} = -\ \frac{\cos A}{\cos C}.$$

By the polar triangle,

$$\frac{d\,A}{d\,C} = -\ \frac{\cos a}{\cos \gamma}.$$

The preceding methods are used when the variation of one part is expressed in the square of the variation of another.

SECTION VIII.

ON GEODETIC OPERATIONS.

152. THE object of Geodetic operations is twofold, namely, the description of a country, such as Great Britain, with respect to its *figure*, *magnitude*, and *position*, on the surface of the globe; and the determination of the dimensions and figure of the earth. The magnitude of a tract of country is found from the circumstance of any one line in it being accurately measured, whilst its position is determined by finding the geographic latitude and longitude of any place within its boundary. But the first thing which is to be done, is to measure with extreme accuracy the length of a line, which is called a *base*, on a plain four or five miles in extent. The principal base being measured, other bases are measured in different situations, for the purpose of being bases of verification, by comparing their measured with their calculated lengths: thus the correctness of the observations is ascertained. In the next place, signals are placed at proper stations, and the country is divided into triangles by arcs of great circles connecting those signals. When the angles which two signals subtend, as seen from a third, are measured either by a repeating circle, in which case it becomes necessary to find the horizontal angle between the signals; or the horizontal angle is de-

termined at once by a theodolite. In the principal triangles the three angles are observed, and the error, if any subsist, is divided among the three angles in the most probable manner: this error, in the more careful observations, has seldom amounted to 2″. In smaller triangles it is sufficient that two angles be measured.

153. To determine the distances of a signal from the extremities of the *base*, and the angle subtended by it, there are given the base, and the two angles observed at its extremities to determine the other parts of the triangle. Similarly, by observing the angles, at the extremities of one of the sides of the triangle just determined, with reference to a new signal, there is a second triangle with similar data: let this process be continued to any number of triangles, until a base of verification be arrived at. The stations are thought to be best chosen, when the sides of the connecting triangles are greater than ten, and less than twenty miles. The sides of the triangles are necessarily very small when compared with the radius of the earth.

154. If it be required to find the length of an arc corresponding to a degree of latitude, the distance of two places in the same meridian must be ascertained, and the latitude of each is to be observed. The determination of the length of a degree of longitude requires the spheroidal form of the earth to be taken in consideration; but it is not intended to enter upon such a subject in this place.

155. Prop. *If the base consist of two parts, as* (a) *and* (b), *forming a very small angle* C, *not greater than* 49′; *it is required to show that the correction*

$$=\frac{ab}{a+b}\,(0.0{,}000{,}000{,}000{,}01175)\,C^2.$$

Let c be the line joining the extremities of the lines denoted by a and b; then

$$c^2 = a^2 + b^2 - 2\,ab \cos(180° - C),$$
$$= a^2 + b^2 + 2\,ab \cos C;$$

but $\cos C = 1 - \frac{1}{2}\left(\frac{\pi}{180°}\right)^2 C^2$ nearly,

$$\therefore\ c^2 = (a+b)^2 - ab\left(\frac{\pi}{180°}\right)^2 C^2,$$

and $c = a + b - \dfrac{ab}{2(a+b)}\left(\dfrac{\pi}{180°}\right)^2 C^2$ nearly,

$$= a + b - \frac{ab}{2(a+b)}\,(0.000004848)^2\,C^2,$$
$$= a + b - \frac{ab}{a+b}\,(0.000{,}000{,}000{,}01175)\,C^2,$$

hence $(a+b) - c =$ the correction is found as given above.

156. Prop. *If* a *be the base of a system of triangles accurately determined in feet; the number of seconds in this base considered as an arc of a great circle of the earth's surface.*

$$= \frac{3600''}{365155}\,a.$$

Let a'' be the number of seconds corresponding to a: then since the number of feet in a degree is 365155, and the number of seconds 3600″,

$$\frac{a''}{3600''} = \frac{a}{365155}$$
$$\therefore\ a'' = 3600''.\ \frac{a}{365155}.$$
$$\text{and } \log a'' = \log 3600 + \log a - \log 365155.$$

157. The measured base being a, the sides b and c are to be found from the equations,

$$\sin b = \sin a . \frac{\sin B}{\sin A},$$

$$\sin c = \sin a . \frac{\sin C}{\sin A};$$

hence it becomes necessary to find the value of $\sin a$.

158. Prop. *The three angles of a plane triangle being observed, and the side a known; it is required to find its figure, when the other two sides are least affected by the errors of observation.*

Let the angle opposite to the known side (a) be observed, and let the errors of the angles be denoted by ΔA, ΔB, ΔC; then, because the sum of the angles, if erroneous, is supposed to be corrected by altering each of the angles by the same quantity,

$$\Delta A = -(\Delta B + \Delta C);$$

and the true equation between a and c is,

$$\begin{aligned}
c &= a . \frac{\sin (C + \Delta C)}{\sin (A + \Delta A)}, \\
&= a . \frac{\sin (C + \Delta C)}{\sin \{A - (\Delta B + \Delta C)\}}, \\
&= a . \frac{\sin C + \cos C . \Delta C}{\sin A - \cos A . \Delta C}, \\
&= a . \left(\frac{\sin C}{\sin A} + \frac{\cos C . \Delta C}{\sin A}\right), \text{ nearly,}
\end{aligned}$$

$$\begin{aligned}
\text{but } \sin B &= \sin (180^\circ - B), \\
&= \sin (A + C), \\
&= \sin A . \cos C + \cos A . \sin C,
\end{aligned}$$

$$\therefore \frac{\cos C}{\sin A} = \frac{\sin B}{(\sin A)^2} - \frac{\cos A . \sin C}{(\sin A)^2},$$

$$\text{hence, } c = a.\left(\frac{\sin C}{\sin A} + \frac{\sin B.\ \Delta C}{(\sin A)^2} - \frac{\cos A.\sin C.\ \Delta C}{(\sin A)^2}\right),$$

or when $\Delta A = 0$, since then $-\ \Delta C = \Delta B$,

$$c = a.\left(\frac{\sin C}{\sin A} + \frac{\sin B.\ \Delta C}{(\sin A)^2} + \frac{\cos A.\sin C.\ \Delta B}{(\sin A)^2}\right),$$

but when there is no error in B and C,

$$c = a.\ \frac{\sin C}{\sin A},$$

$$\therefore \text{ the error of } c = a.\left(\frac{\sin B.\ \Delta C}{(\sin A)^2} + \frac{\cos A.\ \sin C.\ \Delta B}{(\sin A)^2}\right);$$

$$\text{similarly, the error of } b = a.\left(\frac{\sin C.\ \Delta B}{(\sin A)^2} + \frac{\cos A.\sin B.\ \Delta C}{(\sin A)^2}\right).$$

Since, however, the chances of the errors ΔB, ΔC, ΔA, or $-(\Delta B + \Delta C)$, cannot be exactly assigned, all reasoning on them must be vague. Yet it is plain that sin A must not be small, and it is greatest when $A = 90°$. But it is likewise clear, that there is a greater probability that the signs of ΔB and ΔC are different, than that they are the same; since, in the three pairs that can be formed of ΔA, ΔB, ΔC, two will have errors of different signs, and one will have errors of the same sign. If ΔB and ΔC have different signs, the errors of b and c will be diminished, by giving cos A a positive value: A therefore ought to be less than 90°; and if ΔB and ΔC are probably not very different, B and C should be nearly equal. These conditions will be satisfied by a triangle differing not much from an equilateral triangle.

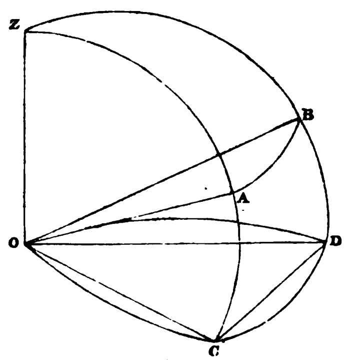
159. Cor. The two angles A and B being observed, the expressions for the errors will be as given above; but there is no reason to think that their errors will have different rather than the same sign: in this case, then, the errors of b and c will probably be the least, if $\cos A = 0$, or $A = 90°$; hence, the

the remaining angle of the triangle ought to be as nearly as possible a right angle.

160. Prop. *Having given the distance* = d *of two objects at different elevations* h *and* h′ *above the horizon, if* Z *be the horizontal angle;*

$$\cos Z = \frac{\cos a - \sin h . \sin h'}{\cos h . \cos h'}.$$

Let OA, OB be the directions in which two objects of different elevations are seen from O; and let the angle AOB be observed.



About O, as centre, let a sphere be described, and through Z, the vertical point to O, draw the great circles ZAC, ZBD, meeting the horizon COD in C and D; draw the straight lines CO, DO: then COD is the horizontal angle = AZB, and Z is the pole of CD.

By (art. 61.)

$$\cos Z = \frac{\cos AB - \cos AZ . \cos BZ}{\sin AZ . \sin BZ},$$

or, by substituting for $AZ = \frac{\pi}{2} - h$, $BZ = \frac{\pi}{2} - h'$; and $AB = d$;

$$\cos Z = \frac{\cos d - \sin h. \sin h'}{\cos h. \cos h'};$$

$$\text{or } \left(\sin \frac{Z}{2}\right)^2 = \frac{\sin\left(\frac{d+h-h'}{2}\right). \sin\left(\frac{d+h'-h}{2}\right)}{\cos h. \cos h'}.$$

161. Prop. *It is required to find the difference between the oblique angle contained between two objects above the horizon, and the horizontal angle.*

Let the angle $Z = d + x$,

$$\text{then, } \cos Z = \cos (d + x),$$

$$= \cos d. \cos x - \sin d. \sin x,$$

$$= \cos d - x. \sin d \text{ nearly,}$$

from the nature of the problem.

$$\text{Hence, } \cos d - x. \sin d = \frac{\cos d - \sin h. \sin h'}{\cos h. \cos h'},$$

$$= \frac{\cos d - hh'}{1-\frac{1}{2}(h^2+h'^2)}, \text{ nearly,}$$

$$= (\cos d - h h'). \{1 - \tfrac{1}{2} (h^2 + h'^2)\}^{-1},$$

$$= (\cos d - hh'). \{1 + \tfrac{1}{2}(h^2 + h'^2)\}, \text{ nearly,}$$

$$= \cos d - h h' + \frac{\cos d}{2} (h^2 + h'^2),$$

$$\therefore x = \frac{h h'}{\sin d} - \frac{\cos d}{2 \sin d} (h^2 + h'^2);$$

let $p = h + h'$, and $q = h - h'$,

$$\therefore \frac{p^2 - q^2}{4} = hh', \text{ and } \frac{p^2 + q^2}{2} = h^2 + h'^2,$$

$$\text{hence, } x = \frac{p^2 - q^2}{4 \sin d} - \frac{\cos d}{4 \sin d} (p^2 + q^2),$$

$$= \tfrac{1}{4}\left\{\frac{1-\cos d}{\sin d}.\ p^2 - \left(\frac{1+\cos d}{\sin d}\right).\ q^2\right\},$$

$$= \tfrac{1}{4}\left(p^2.\ \tan\frac{d}{2} - q^2.\ \cot\frac{d}{2}\right).$$

For x, if it contain m seconds, the quantity m (0.00004848) must be used.

This is called the approximate reduction to the horizon, and was first given by *Legendre;* but, for observations with the theodolite, it is not necessary.

162. The horizontal angles being found by the above method, the triangles become spherical, and their sides are very small when compared with the radius of the sphere. These triangles are solved by *three* different methods.

First Method.

The triangles being considered spherical, since the sides are very small when compared with the radius:

$$\frac{\sin a}{r} = \frac{a}{r} - \frac{a^3}{6r^3},\ \text{very nearly},$$

$$= \frac{a}{r}\left(1 - \frac{a^2}{6r^2}\right);$$

but $\dfrac{\cos a}{r} = 1 - \dfrac{a^2}{2r^2}$,

and $\left(1-\dfrac{a^2}{2r^2}\right)^{\frac{1}{3}} = 1 - \dfrac{a^2}{6r^2}$, nearly;

hence, $\dfrac{\sin a}{r} = \dfrac{a}{r}.\ \left(\dfrac{\cos a}{r}\right)^{\frac{1}{3}}$,

or $\sin a = a.\left(\dfrac{\cos a}{r}\right)^{\frac{1}{3}}$,

$$= a. (\cos a'')^{\frac{1}{3}};$$

the value of a'' being that which is stated in (art. 154.)

Since a is given in feet or toises, log a must be found from a table where a is expressed in feet or toises: thus, because

$$\log \sin a = \log a + \tfrac{1}{3} \log \cos a'',$$

sin a is found in feet, hence,

$$\sin b = \sin a. \frac{\sin \mathrm{B}}{\sin \mathrm{A}},$$

$$\text{and } \sin c = \sin a. \frac{\sin \mathrm{C}}{\sin \mathrm{A}},$$

are likewise found in feet.

This method is preferred by *Delambre* to the two following.

Second Method.

163. This is to find from the angles of the spherical triangles the angles included between their chords, and then to solve the triangle as if it were plane.

Let C be the spherical angle,

and C′ the angle between the chords,

then, by (art. 127.)

$$\cos \mathrm{C}' = \cos \frac{\alpha}{2}. \cos \frac{\beta}{2}. \cos \mathrm{C} + \sin \frac{\alpha}{2}. \sin \frac{\beta}{2};$$

suppose $\mathrm{C}' = \mathrm{C} - x$, where x is small,

$$\text{then } \cos \mathrm{C}' = \cos (\mathrm{C} - x) = \cos \mathrm{C} + \sin \mathrm{C}. x,$$

$$= \cos \frac{\alpha}{2}. \cos \frac{\beta}{2}. \cos \mathrm{C} + \sin \frac{\alpha}{2}. \sin \frac{\beta}{2},$$

$$\text{hence, } x \sin \mathrm{C} = \sin \frac{\alpha}{2}. \sin \frac{\beta}{2} - \left(1 - \cos \frac{\alpha}{2}. \cos \frac{\beta}{2}\right). \cos \mathrm{C},$$

$$= \frac{\alpha\beta}{4} - \frac{\alpha^2 + \beta^2}{8}. \cos \mathrm{C};$$

but $e = \alpha + \beta$, and $f = \alpha - \beta$,

$$\text{then } \alpha\beta = \frac{e^2 - f^2}{4}, \text{ and } \alpha^2 + \beta^2 = \frac{e^2 + f^2}{2},$$

$$\therefore x \sin C = \frac{e^2 - f^2}{16} - \frac{e^2 + f^2}{16} \cdot \cos C,$$

$$\therefore x = \frac{1}{16}\left\{e^2\left(\frac{1 - \cos C}{\sin C}\right) - f^2\left(\frac{1 + \cos C}{\sin C}\right)\right\},$$

$$= \frac{1}{16}\left(e^2 \tan\frac{C}{2} - f^2 \cot\frac{C}{2}\right).$$

In these expressions the angles are expressed in numbers, rad $= 1$; if $e = n$ feet, instead of e the quantity $\dfrac{n}{\text{numb. of feet in rad}}$ must be used; and for x, if it contain m seconds, the quantity $m \times 0.000004848$. This method was used in the English surveys.

Third Method.

164. *It is required to find the quantity by which the spherical angles must be diminished, in order that the triangle may be treated as a plane one.*

Let the sides to radius r be a, b, c; then to rad $= 1$, the sides of a similar triangle are $\frac{a}{r}$, $\frac{b}{r}$, $\frac{c}{r}$; and

$$\frac{1 + \cos A}{2} = \frac{\sin\frac{S}{r} \cdot \sin\frac{S - a}{r}}{\sin\frac{b}{r} \cdot \sin\frac{c}{r}},$$

$$= \left(\frac{S}{r} - \frac{S^3}{6r^3}\right)\left(\frac{S-a}{r} - \frac{(S-a)^3}{6r^3}\right)\left(\frac{b}{r} - \frac{b^3}{6r^3}\right)^{-1}\left(\frac{c}{r} - \frac{c^3}{6r^3}\right)^{-1},$$

$$= \frac{S(S-a)}{bc}\left(1 - \frac{S^2}{6r^2}\right)\left(1 - \frac{(S-a)^2}{6r^2}\right)\left(1 + \frac{b^2}{6r^2}\right)\left(1 + \frac{c^2}{6r^2}\right),$$

$$=\frac{\text{S}.(\text{S}-a)}{b\,c}\left(1-\frac{\text{S}^2}{6r^2}-\frac{(\text{S}-a)^2}{6r^2}\right)\left(1+\frac{b^2+c^2}{6r^2}\right),$$

$$=\frac{\text{S}.(\text{S}-a)}{b\,c}\left(1+\frac{b^2+c^2}{6r^2}-\frac{\text{S}^2}{6r^2}-\frac{(\text{S}-a)^2}{6r^2}\right), \text{ nearly,}$$

$$=\frac{\text{S}.(\text{S}-a)}{b\,c}-\frac{\text{S}.(\text{S}-a)}{b\,c}\left(\frac{\text{S}^2+(\text{S}-a)^2-b^2-c^2}{6\,r^2}\right),$$

$$=\frac{\text{S}.(\text{S}-a)}{bc}-\frac{\text{S}.(\text{S}-a)}{bc}\left(\frac{a^2-(b-c)^2}{12r^2}\right),$$

$$=\frac{\text{S}.(\text{S}-a)}{b\,c}-\frac{b\,c}{12\,r^2}\cdot\frac{4}{(bc)^2}\sin\text{S}.\sin(\text{S}-a).\sin(\text{S}-b).\sin(\text{S}-c).$$

Let A′, B′, C′, be the angles of the plane triangle,

$$\therefore\frac{1+\cos\text{A}}{2}=\frac{1+\cos\text{A}'}{2}-\frac{bc}{12\,r^2}(\sin\text{A}')^2,$$

$$\therefore\ \cos\text{A}=\cos\text{A}'-\frac{bc}{6\,r^2}(\sin\text{A}')^2\,;$$

$$=\cos(\text{A}'+x),\text{ suppose,}$$

$$=\cos\text{A}'-\frac{\pi x}{180^\circ}.\sin\text{A}',\text{ nearly};$$

$$\text{hence, }\frac{\pi x}{180^\circ}=\frac{bc}{6\,r^2}.\sin\text{A}',$$

$$=\frac{1}{3r^2}.\frac{bc}{2}.\sin\text{A}',$$

$$=\frac{1}{3r^2}\times\text{area of plane triangle,}$$

$$\therefore\ x(0.000004848=\frac{a}{3\,r^2},\text{ suppose,}$$

hence, if x contain (0.000004848) n times,

$$n''=\frac{a}{3\,r^2\times 0.000004848}.$$

This correction is the same for each of the angles;

$$\therefore\ A' = A - x,$$
$$B' = B - x,$$
$$C' = C - x,$$
$$\text{and } 3x = A + B + C - 180^\circ = E,$$
$$\therefore\ x = \frac{E}{3}.$$

Hence, the proposed triangle may be treated as a plane one, when its spherical angles are diminished by an angle equal to *one third* of the spherical excess, which seldom exceeds 5″. This Theorem is likewise due to *Legendre.*

165. The area of the triangle being found in feet,

$$\log (3\, r^2 \times 0.000004848) = 9,8038940,$$

the number of feet in a degree on the earth's surface being estimated at 365155 feet. The area of the triangle can always be found with sufficient exactness from its parts, which are known accurately enough for practice.

166. This method is not confined to triangles, the sides of which are very small; it may be applied even to cases, where the sides exceed a degree and a half, with sufficient accuracy. It is likewise applicable to triangles described on spheroids of small eccentricity.

167. Since, in article 162,

$$a = \frac{bc}{2} \sin A',$$

$$\text{and that, } b = a.\frac{\sin B'}{\sin (B' + C')}; \text{ for } \sin (B' + C') = \sin A',$$

$$c = a.\frac{\sin C'}{\sin (B' + C')},$$

$$\therefore\ a = \frac{a^2}{2}.\frac{\sin B'.\sin C'}{\sin (B' + C')}, \text{ in terms of the observed angles.}$$

168. PROP. *The area of the triangle being given in square feet, it is required to find from it the spherical excess in seconds.*

Let a be the area of the triangle to rad $= r$, the earth's radius; then,

$$\text{E in seconds} = \frac{180.\,60.\,60}{\pi}.\ \frac{a}{r^2},\ \text{(art. 57.)}$$

$$\text{but } 1^\circ = \frac{180^\circ}{\pi}.\ \frac{\text{length of degree in feet}}{r};$$

or, in seconds,

$$60.\,60 = \frac{180.\,60.\,60.}{\pi}\ \frac{(60859.1)\,6}{r},$$

$$\therefore\ r = \frac{180.\,60.\,60}{\pi} \times \frac{60859.\,1}{600},$$

$$\therefore\ \text{E} = \frac{\pi}{180.\,60.\,60}.\ \frac{(600)^2 a}{(60859.\,1)^2},$$

$$= \frac{(600)^2 a}{206265.\,(60859.1)^2},$$

$$\text{hence,} \log \text{E} = \log a - \log\left(\frac{60859.\,1}{(600)^2} \times 206265\right),$$

$$= \log a - 9.\,3267737.$$

Thus, if $\log a = 8.\,5026328$,

$$\log \text{E} = -\ 1.1758591,$$

the number corresponding to which is

$$.14992,\ \text{or E} = 0''.\,15.$$

SECTION IX.

ON REGULAR POLYHEDRONS, &c.

169. Prop. *If* F = *number of regular faces,*
E = *number of edges,*
S = *number of solid angles,*
n = *number of sides in a face,*

$$\textit{then, } nF = 2E, \textit{ or } n = \frac{2E}{F}, \quad (1.)$$

$$\textit{and } F + S = E + 2. \quad (2.)$$

(1.) Since every edge is made by *two* sides, therefore the whole number of sides in the polyhedron* is equal to $2E$; but this number is also equal to nF,

$$\therefore nF = 2E,$$

$$\text{or } n = \frac{2E}{F}.$$

(2.) Take any point within the polyhedron, and from it draw lines to all the angular points; then, if a spherical sur-

---

* For the definitions of the regular polyhedrons, &c. the student is referred to Euc. XI. Def.

face (rad $= 1$) be supposed to be described about this point, cutting the lines just drawn, and the points of intersection be joined so as to form as many spherical polygons as there are faces in the polyhedron; the area of one of these polygons

$$= \frac{\pi}{180^\circ} \{\text{angles of polygon} - (n-2)\,180^\circ\};$$

hence, the area of all these polygons, or

$$\frac{\pi}{180^\circ} \{\text{F} \times \text{angles of polygon} - \text{F}\,(n-2)\,180^\circ\} = 4\,\pi,$$

which is the surface of the sphere rad $= 1$. Now, it has been shown, that $n\text{F} = 2\text{E}$, and all the spherical angles that can be formed about a point on the surface of a sphere $= 360^\circ$, therefore $\text{F} \times$ angles of polygon $= \text{S}.\,360^\circ$;

$$\text{hence, } \frac{\pi}{180^\circ} \{\text{S}.\,360^\circ - (2\,\text{E} - 2\,\text{F})\,180^\circ\} = 4\,\pi,$$

$$\text{or } \text{S} - \text{E} + \text{F} = 2,$$

$$\therefore \text{S} + \text{F} = \text{E} + 2.$$

It is manifest this equation is also true when the polyhedron is not regular.

170. Cor. $\text{E} - \text{F} = \text{S} - 2$.

171. Prop. *The sum of all the plane angles, which form the solid angles of a polyhedron*

$$= (\text{S} - 2)\,360^\circ.$$

For all the interior angles of one of the faces of (n) sides $=$ $(n - 2)\,180^\circ$, therefore the sum of all the plane angles of all the plane faces $= \text{F}\,(n - 2)\,180^\circ$,

$$= (\text{F}\,n - 2\,\text{F})\,180^\circ,$$

$$= (\text{E} - \text{F})\,360^\circ,$$

$$= (\text{S} - 2)\,360^\circ.$$

172. PROP. *If* n = *number of sides in a plane face,* m = *number of plane angles containing a solid angle;*

$$then\ S = \frac{4n}{2(m+n)-mn},$$

$$F = \frac{4m}{2(m+n)-mn},$$

$$E = \frac{mn2}{2(m+n)-mn}.$$

For each plane angle of the solid angle

$$= \left(\frac{n-2}{n}\right) 180°;$$

hence the sum of all the plane angles, which form one solid angle $= m. \left(\frac{n-2}{n}\right) 180°$, and, therefore, the sum of all the plane angles which form all the solid angles $= m. \left(\frac{n-2}{n}\right)$ S. 180°,

$$\text{hence, } m\left(\frac{n-2}{n}\right) S.\ 180° = (S-2)\ 360°, \text{ (art. 169.)}$$

$$\therefore\ mn\,S - 2m\,S = 2n\,S - 4n,$$

$$\text{hence } \{2(m+n) - mn\}\ S = 4n,$$

$$\therefore\ S = \frac{4n}{2(m+n)-mn}. \qquad (1.)$$

$$\text{Also, } F = E + 2 - S,$$

$$= \frac{nF}{2} + 2 - S,$$

$$= \frac{nF}{2} + \frac{4m - 2mn}{2(m+n)-mn},$$

$$\therefore\ \frac{n-2}{2}.\ F = \frac{2m(n-2)}{2(m+n)-mn},$$

$$\text{and } F = \frac{4m}{2(m+n)-mn}. \qquad (2.)$$

And, since $E + 2 = S + F$; there results,

$$E = \frac{4m + 4n}{2(m+n) - mn} - 2,$$
$$= \frac{2mn}{2(m+n) - mn}. \qquad (3.)$$

173. Cor. $\frac{S}{F} = \frac{n}{m}$; $\frac{E}{S} = \frac{m}{2}$;

and $\frac{E}{F} = \frac{n}{2}$.

174. Prop. *There can be only five regular polyhedrons, viz. the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron.*

Since S, F, and E, are by the supposition positive integers, their values as found in (art. 170.) must also be positive integers; hence $2(m+n)$ must be greater than mn, or $\frac{m+n}{m}$ must be greater than $\frac{n}{2}$.

(1.) Now the least values of m and n are 3 and 3, in which case $\frac{m+n}{m}$ is greater than $\frac{n}{2}$; and the values of S, F, and E are 4, 4, and 6 respectively: hence the solid is a regular *tetrahedron*, or pyramid.

(2.) Let $m = 3$, and $n = 4$: then $\frac{m+n}{m}$ is greater than $\frac{n}{2}$; and $S=8$, $F=6$, and $E=12$; thence the solid, in this case, is a *hexahedron*, which is the same as the cube.

(3.) Let $m = 4$, and $n = 3$: then $\frac{m+n}{m}$ is greater than $\frac{n}{2}$; and $S = 6$, $F = 8$, and $E = 12$. Or, the solid has eight plane triangular faces, for which reason it is called an *octahedron*.

(4.) If $m = 3$, and $n = 5$, the condition of $\frac{m+n}{m}$ being greater than $\frac{n}{2}$ is still fulfilled; and $S = 20$, $F = 12$, and $E = 30$. In this case the solid is a *dodecahedron*. But if m be still $= 3$, and n be any number greater than 5, the requisite condition is no longer satisfied; which circumstance shows, that there can be only three regular polyhedrons, when $m = 3$. If $m = 4$, and n also $= 4$, or any greater number, $\frac{m+n}{m}$ is not greater than $\frac{n}{2}$; hence there can be only one regular polyhedron, viz. the *octahedron*, when $m = 4$.

(5.) If $m = 5$, and $n = 3$, the requisite condition is fulfilled, and $S = 12$, $F = 20$, $E = 30$. And the solid is an *icosahedron*.

But if m be 5, or any number greater than 5, and n be any number greater than 3, the requisite condition ceases to exist. From the above remarks, it is concluded that there are five, and not more than five, regular polyhedrons.

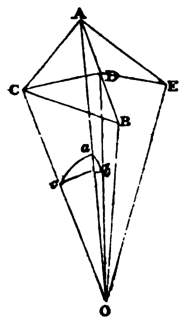
175. Let $2(m + n) - m\,n = 0$, then S, F, and E, each equal infinity; hence, the solid in this case must be a sphere, the surface of which is supposed to contain an infinite number of regular plane faces.

176. Prop.

Let n = *number of sides in a face of the polyhedron,*
m = *number of plane angles in a solid angle,*
I = *inclination of two contiguous faces;*

$$\text{then, } \sin\frac{I}{2} = \frac{\cos\frac{180^\circ}{m}}{\sin\frac{180^\circ}{n}}.$$

Let AB be an edge of the polyhedron, D its bisection, C the centre of one of the faces, E the centre of an adjacent



face; join CD, DE, which are at right angles to AB: then the angle CDE measures the inclination of the two contiguous faces. In the plane in which CD and DE are, let CO and EO be drawn at right angles to CD, DE meeting in O; join OA, OB, OD; about O, as a centre, describe a spherical surface cutting OA, OD, OC, in a, b, c, which are supposed to be connected by arcs of great circles; then, because the plane COE is perpendicular to the plane AOB, the angle $a\,b\,c$ is a right angle; also, the angle

$$c\,a\,b = \frac{360^\circ}{2m} = \frac{180^\circ}{m},$$

$$\text{and angle } a\,c\,b = \frac{360^\circ}{2n} = \frac{180^\circ}{n};$$

hence, by *Napier's* rules,

$$\cos c\,a\,b = \cos c\,b.\ \sin a\,c\,b;$$

$$\text{but } \cos\ c\,b = \cos \text{COD} = \cos\left(90^\circ - \frac{\text{I}}{2}\right) = \sin\frac{\text{I}}{2},$$

$$\therefore \sin\frac{\text{I}}{2} = \frac{\cos c\,a\,b}{\sin a\,c\,b} = \frac{\cos\dfrac{180^\circ}{m}}{\sin\dfrac{180^\circ}{n}};$$

hence, in the five regular polyhedrons, the values of m and n being given, the inclination of two contiguous faces can be found.

177. Cor. Since, from (art. 171.),

$$m = \frac{2\,E}{S}, \text{ and } n = \frac{2\,E}{F};$$

$$\therefore \sin\frac{I}{2} = \frac{\cos\frac{S}{E}\,90^\circ}{\sin\frac{F}{E}\,90^\circ}.$$

178. Prob. *To find the value of* I *in the five regular polyhedrons.*

(1.) In the tetrahedron, $m = 3$, and $n = 3$;

$$\therefore \sin\frac{I}{2} = \frac{\cos 60^\circ}{\sin 60^\circ} = \frac{1}{\sqrt{3}}; \text{ hence, } I = \cos^{-1}\frac{1}{3}.$$

(2.) In the hexahedron, $m = 3$, and $n = 4$;

$$\therefore \sin\frac{I}{2} = \frac{\cos 60^\circ}{\sin 45^\circ} = \frac{1}{\sqrt{2}}; \text{ hence, } I = 90^\circ.$$

(3.) In the octahedron, $m = 4$, and $n = 3$;

$$\therefore \sin\frac{I}{2} = \frac{\cos 45^\circ}{\sin 60^\circ} = \sqrt{\frac{2}{3}}; \text{ hence, } I = \cos^{-1}\left(-\frac{1}{3}\right).$$

(4.) In the dodecahedron, $m = 3$, and $n = 5$;

$$\therefore \sin\frac{I}{2} = \frac{\cos 60^\circ}{\sin 36^\circ} = \frac{2}{\sqrt{(10-2\sqrt{5})}}; \text{ hence, } I = \cos^{-1}\left(\frac{1-\sqrt{5}}{5-\sqrt{5}}\right).$$

(5.) In the icosahedron, $m = 5$, and $n = 3$;

$$\therefore \sin\frac{I}{2} = \frac{\cos 36^\circ}{\sin 60^\circ} = \frac{1+\sqrt{5}}{2\sqrt{3}}; \text{ hence, } I = \cos^{-1}\left(-\frac{\sqrt{5}}{3}\right).$$

179. PROB. *To find the radius of the sphere inscribed in a regular polyhedron.*

From the construction of the figure in (art. 176.), it is plain that OC = OE is the radius of the sphere required: let OC $= r$, and AB $= 2l$.

$$\text{Now, the angle ACD} = \tfrac{1}{2}.\frac{360^\circ}{n} = \frac{180^\circ}{n},$$

$$\therefore\ l = \text{CD}.\tan\frac{180^\circ}{n},$$

$$\text{or, CD} = l.\cot\frac{180^\circ}{n};$$

$$\text{hence, CD} = \text{CO}.\tan \text{COD}$$

$$= r.\ \tan\left(90^\circ - \frac{\text{I}}{2}\right)$$

$$= r.\ \cot\frac{\text{I}}{2} = l.\ \cot\frac{180^\circ}{n},$$

$$\therefore\ r = l.\tan\frac{\text{I}}{2}.\ \cot\frac{180^\circ}{n}.$$

If the solid be given, I and n are known; therefore, the magnitude of r can be obtained.

180. PROB. *To find the radius of the sphere circumscribed about a regular polyhedron.*

From figure in (art. 176.), it is obvious that the radius of the sphere required is equal to OA, or OB. Let OA = R; and, from the last article,

$$\text{CD} = l.\ \cot\frac{180^\circ}{n} = \text{AC}.\cos\frac{180^\circ}{n},$$

$$\therefore\ \text{AC} = l.\ \text{cosec}\frac{180^\circ}{n};$$

$$\text{hence, R}^2 = \text{OA}^2 = \text{OC}^2 + \text{AC}^2,$$

and $R^2 - r^2 = l^2 \left(\text{cosec}\,\frac{180°}{n}\right)^2$;

also, $R^2 = l^2 \left(\tan\frac{I}{2}\right)^2 \left(\cot\,\frac{180°}{n}\right)^2 + l^2 \left(\text{cosec}\,\frac{180°}{n}\right)^2$;

$$= l^2 \left\{ \frac{\left(\cos\,\frac{180°}{n}\right)^2 \left(\cot\,\frac{180°}{n}\right)^2}{\left(\sin\frac{180°}{n}\right)^2 - \left(\cos\,\frac{180°}{m}\right)^2} + \left(\text{cosec}\,\frac{180°}{n}\right)^2 \right\},$$

$$= l^2 \left\{ \frac{\left(\cos\frac{180°}{m}\right)^2 \left(\text{cosec}\,\frac{180°}{n}\right)^2 - \left(\cos\frac{180°}{m}\right)^2}{\left(\sin\,\frac{180°}{n}\right)^2 - \left(\cos\,\frac{180°}{m}\right)^2} + \left(\text{cosec}\,\frac{180°}{n}\right)^2 \right\},$$

$$= l^2 \left\{ \frac{1 - \left(\cos\,\frac{180°}{m}\right)^2}{\left(\sin\,\frac{180°}{n}\right)^2 - \left(\cos\,\frac{180°}{m}\right)^2} \right\},$$

$$= \frac{l^2 \left(\sin\,\frac{180°}{m}\right)^2}{\left(\sin\,\frac{180°}{n}\right)^2 - \left(\cos\,\frac{180°}{m}\right)^2},$$

$$\therefore R = \frac{l\,\sin\,\frac{180°}{m}}{\sqrt{\left\{\left(\sin\frac{180°}{n}\right)^2 - \left(\cos\frac{180°}{m}\right)^2\right\}}}.$$

181. Prop. $\frac{R}{r} = \tan\,\frac{180°}{m}.\,\tan\,\frac{180°}{n}$,

and $R = l.\,\tan\,\frac{I}{2}.\,\tan\,\frac{180°}{m}$.

Since $\frac{OC}{OA} = \frac{r}{R} = \cos AOC$, from fig. in (art. 176.)

$= \cos ac = \cot cab.\,\cot acb$,

hence $\frac{R}{r} = \tan\frac{180°}{m}.\ \tan\frac{180°}{n}$;

also $R = r.\ \tan\frac{180°}{m}.\ \tan\frac{180°}{n}$,

$= l.\tan\frac{I}{2}.\ \tan\frac{180°}{m}$. (art. 177.)

182. PROB. *To find the values of* r *and* R *in the five regular polyhedrons.*

(1.) In the tetrahedron, since $m = 3 = n$;

$$\frac{R}{r} = (\tan 60)^2 = 3,$$

$$\therefore\ R = 3r:$$

also $R^2 - r^2 = l^2 (\text{cosec } 60)^2 = \frac{4l^2}{3}$; (art. 178);

hence $\frac{8\ R^2}{9} = \frac{4l^2}{3}$,

$$\therefore\ R = \left(\frac{3}{2}\right) l, \text{ and } r = \frac{l}{\sqrt{6}}.$$

(2.) and (3.) In the hexahedron, since $m = 3$ and $n = 4$, and in the octahedron, $m = 4$ and $n = 3$;

$$\therefore \frac{R}{r} = \tan 60°.\ \tan 45° = \sqrt{3}, \text{ in both cases;}$$

also $R^2 - r^2 = l^2 (\text{cosec } 45°)^2$, when $n = 4$;

$$= 2\,l^2,$$

$$\therefore\ R^2 = 3\,l^2,$$

and $R = l\sqrt{3},\ r = l$:

when $n = 3$, in the octahedron;

$$R^2 - r^2 = l^2 (\text{cosec } 60°)^2,$$

$$= \frac{4l^2}{3},$$

$$\therefore\ R = l\sqrt{2}, \text{ and } r = \sqrt{\left(\frac{2}{3}\right)}\ l.$$

(4.) and (5.) In the dodecahedron, $m = 3$, and $n = 5$; in the icosahedron, $m = 5$, and $n = 3$; hence, in both these cases,

$$\frac{R}{r} = \tan 60°. \tan 36°,$$

$$= \sqrt{\left(\frac{5 + 2\sqrt{5}}{15}\right)};$$

also $R^2 - r^2 = l^2 (\operatorname{cosec} 36°)^2$, $n = 5$,

$$= \frac{5-\sqrt{5}}{8} l^2;$$

$$\text{hence } R = \left(\frac{\sqrt{15} + \sqrt{5}}{2}\right) l, \text{ and } r = \frac{\sqrt{(250-110\sqrt{5})}}{10} l:$$

lastly, $R^2 - r^2 = l^2 (\operatorname{cosec} 60°)^2$, $n = 3$,

$$= \frac{4l^2}{3};$$

$$R = \frac{\sqrt{(10+2\sqrt{5})}}{2} l,$$

$$\text{and } r = \frac{\sqrt{(42 + 18\sqrt{5})}}{6} l.$$

The values of $\frac{R}{r}$ in the four last polyhedrons show, that if the two pairs of solids were respectively inscribed in a sphere, each pair might be circumscribed about another sphere, and the contrary.

183. Prop. *The content of a regular polyhedron*

$$= \frac{F n l^3 \tan \frac{I}{2}}{\left(\tan \frac{180°}{n}\right)^2} : \textit{where } \tan\frac{I}{2} = \sqrt{\frac{\cos \frac{180°}{m}}{\left\{\left(\sin\frac{180°}{n}\right)^2 - \left(\cos\frac{180°}{m}\right)^2\right\}}}.$$

In the fig. (art. 176.) from O the centre of the inscribed sphere, draw OA, OB, &c. to all the angular points of the polyhedron, which will consequently be divided into as many pyramids as there are regular faces in the solid, the radius (r)

of the inscribed sphere being the altitude of each pyramid: hence, if a be the area of each plane face, the content required

$$=\frac{\mathrm{F}ar}{3},$$

$$\text{where } a=\frac{rl^2}{\tan\frac{180^\circ}{n}}, \text{ (art. 315. pt. I.)}$$

$$\text{and } r=l.\tan\frac{\mathrm{I}}{2}.\ \cot\frac{180^\circ}{n}; \text{ (art. 177.)}$$

hence, by substitution,

$$\text{the content}=\frac{\mathrm{F}\,nl^3\tan\frac{\mathrm{I}}{2}}{\left(\tan\frac{180^\circ}{n}\right)^2};$$

$$\propto l^3,$$

in polyhedrons of the same kind.

184. By substituting in the expression $\frac{\mathrm{F}ar}{3}$ the results already obtained; C_1, C_2, &c. representing the contents of the tetrahedron, hexahedron, &c.

$$\mathrm{C}_1=\frac{2\sqrt{2}}{3}l^3.\ \ \mathrm{C}_2=(2\,l)^3.\ \ \mathrm{C}_3=\frac{16}{3\sqrt{2}}l^3.$$

$$\mathrm{C}_4=2\sqrt{(470-210\sqrt{5})}\,l^3.$$

$$\mathrm{C}_5=\frac{10}{3}\sqrt{(14+6\sqrt{5})}\,l^3.$$

185. PROP. *The number of faces, which have an odd number of sides, is always even.*

The same notation remaining,

let f_3 = number of triangular faces,
f_4 = ... quadrilateral . . .,
f_5 = ... pentagonal . . . .,
&c. = &c.

then $F = f_3 + f_4 + f_5 + f_6 + f_7 \ldots = \Sigma f_n$,

and the whole number of sides,

which $= 2E = 3f_3 + 4f_4 + 5f_5 + \ldots = \Sigma n f_n$;

hence $2E - 2F = f_3 + 2f_4 + 3f_5 + \ldots = \Sigma(n-2)f_n$,

but $F + S = E + 2$, (art. 169.)

$\therefore 2S = 4 + 2E - 2F,$

$= 4 + \Sigma(n-2)f_n;$

which shows that $\Sigma(n-2)f_n$ denotes an even number,

$\therefore f_3 + 2f_4 + 3f_5 + 4f_6 + 5f_7 + \ldots =$ even number,

or $(f_3 + f_5 + f_7 + \ldots) + 2(f_4 + f_5 + 2f_6 + 2f_7 +) =$ even numb.

$\therefore f_3 + f_5 + f_7 + \ldots =$ difference of two even numbers is likewise even.

186. Cor. 1. Since the least value of n is 3, the least value of $\Sigma f_n = f_3$, in which case $F = f_3$: in the case of the three regular polyhedrons with triangular faces the even values of f_3 are 4, 8, and 20; as has been seen already.

187. Cor. 2. $\Sigma n f_n$ cannot be less than $3f_3$; that is, $2E$ cannot be less than $3F$; but $2E$ may equal $3F$, as is the case in the regular tetrahedron. Also, since $2S = 4 + \Sigma f_3$, when n has its least value, it follows that S cannot be less than $2 + \frac{F}{2}$, but S may equal this quantity.

188. Prop. *There cannot be a polyhedron, all of whose faces have each more than five sides.*

Since no solid angle can be contained by fewer than three plane angles,

$$\frac{2E}{S} \text{ cannot be less than } 3,$$

$$\text{or } 4E \ldots\ldots\ldots\ldots 6S,$$

$\therefore \Sigma\, 2\, n\, f_n$ cannot be less than $12 + \Sigma\,(3\,n - 6)\,f_n$, (art. 183.);
hence

$$\Sigma\,(6-n)\,f_n \text{ cannot be less than } 12;$$

that is, by giving n its successive values of 3, 4, 5, &c.

$3f_3 + 2f_4 + f_5 - f_7 - 2f_8 - 3f_9 - \ldots$ cannot be less than 12;
or $3f_3 + 2f_4 + f_5$ cannot be less than $12 + f_7 + 2f_8 + 3f_9 + \ldots$

Now, if there were no faces in the polyhedron with three, four, or five sides, $3f_3 + 2f_4 + f_5$ would $= 0$; hence it is concluded, that there cannot exist a polyhedron with faces which have more than five sides in every face.

189. PROP. *In all polyhedrons of whatever kind;*

$$2\,\text{F} \textit{ is not less than } \text{S} + 4,$$
$$\textit{and } 3\,\text{F} \;\ldots\ldots\ldots\; \text{E} + 6.$$

$$\begin{aligned}
&\text{Since } \Sigma\,(6-n)\,f_n \text{ cannot be less than } 12,\\
&\therefore 6\,\Sigma f_n - \Sigma\, n f_n \;\ldots\ldots\ldots\ldots\; 12,\\
&\text{or } 6\,\text{F} - 2\,\text{E} \;\ldots\ldots\ldots\ldots\; 12, \text{(art. 185.)}\\
&\text{hence } 3\,\text{F} \;\ldots\ldots\; \text{E} + 6;
\end{aligned}$$

again, because 3 F is not less than E + 6,

$$\begin{aligned}
&\text{that is, } 3\,\text{F} \;\ldots\ldots\ldots\; 4 + (\text{E} + 2),\\
&\therefore 3\,\text{F} \;\ldots\ldots\ldots\; 4 + \text{S} + \text{F},\\
&\text{hence } 2\,\text{F} \;\ldots\ldots\ldots\; \text{S} + 4.
\end{aligned}$$

190. PROP. *If* $\dfrac{2\,\text{E}}{\text{S}}$, *which denotes the mean number of plane angles forming each solid angle, be greater than* 4; *then*

$$\text{E} \textit{ is not greater than } 2\,\text{F} - 4,$$
$$\textit{and } \text{S} \;\ldots\ldots\ldots\ldots\; \text{F} - 2;$$

where the lowest limit of F *is* 8.

For, since 2 E is greater than 4 S,

$$\begin{aligned}
&\therefore 2\,\text{E} \;\ldots\ldots\ldots\; 4\,(\text{E} + 2 - \text{F}),\\
&\text{or } 2\,\text{F} - 4 \;\ldots\ldots\ldots\; \text{E},
\end{aligned}$$

$\therefore$ E is not greater than $2\,F - 4$;

again, $4\,S$ is less than $2\,E$,

$\therefore$ $4\,S$ is not greater than $2\,(F + S - 2)$,

or $2\,S$ . . . . $2\,F - 4$,

$\therefore$ S . . . . $F - 2$.

Since $2\,E$ is greater than $4\,S$,

$\Sigma\, n f_n$ is not less than $8 + \Sigma\,(2n - a) f_n$, (art 185.)

Hence, by making $n = 3, 4, 5$, &c. successively,

$3f_3 + 4f_4 + 5f_5 + \ldots$ is not less than $8 + 2f_3 + 4f_4 + 6f_5 + 8f_6 + \ldots$

$\therefore f_3$ . . . . $8 + f_5 + 2f_6 + 3f_7 + \ldots$

this shows that the solid must have at least 8 triangular faces, on the supposition that $2\,E$ is greater than $4\,S$. Hence, the lowest limit of F being determined, the lowest limits of S and E are found.

191. Prop. *If* $\dfrac{2\,E}{S}$ *be greater than* 5;

E *is not greater than* $\dfrac{5}{3}(F - 2)$,

and S . . . . $\dfrac{2}{3}(F - 2)$,

where the lowest limit of F *is* 20.

For, since $2\,E$ is greater than $5\,S$,

$\therefore 2\,E$ . . . . $5\,(E + 2 - F)$,

or $5\,(F + 2)$ . . . . $3\,E$,

$\therefore$ E is not greater than $\dfrac{5}{3}(F - 2)$;

again, $5\,S$ is less than $2\,E$,

$\therefore$ $5\,S$ is not greater than $2\,(S + F - 2)$,

$\therefore 3\,S$ . . . . $2\,(F - 2)$,

and S . . . . $\dfrac{2}{3}(F - 2)$.

Since 2 E is greater than 5 S,

$\therefore \Sigma\, n f_n$ is not less than $10 + \Sigma \left(\frac{5n}{2} - 5\right) f_n$, (art. 185.)

hence $3f_3+4f_4+5f_5+\ldots$ is not less than $10+\frac{5}{2}f_3 + 5f_4 + \frac{15}{2}f_5 + \ldots$

$\therefore\ 6f_3+8f_4+10f_5+ \ldots\ldots\ldots\ldots\ 20 + 5f_3+10f_4+15f_5+\ldots$

$\therefore\ f_3\ \ldots\ldots\ldots\ 20+2f_4+5f_5+8f_6+\ldots$

Hence the solid must have at least 20 triangular faces, and the lowest limits of E and S in terms of F have been determined.

192. Prop. *There cannot be a polyhedron, which has all its solid angles formed of six or more plane angles.*

It has been observed, that the mean number of plane angles, composing a solid angle, is denoted by $\frac{2\,E}{S}$. And by (art. 185)

$$2\,S = 4 + \Sigma\,(n-2) f_n,$$
$$= 4 + f_3+2f_4+3f_5+4f_6+5f_7 + \ldots$$

by giving to n its values of 3, 4, 5, &c.

$$\therefore 6\,S = 12 + 3f_3 + 6f_4 + 9f_5 + 12f_6 + \ldots$$
$$= 12 + 2\,E + 2f_4 + 4f_5 + 6f_6 + 8f_7 + \ldots$$
$$\therefore \frac{2\,E}{S} = 6 - \frac{12}{S} - \frac{2}{S}(f_4 + 2f_5 + 3f_6 + \ldots);$$

hence the mean value of $\frac{2\,E}{S}$ is always less than 6, and the truth of the proposition becomes evident. This agrees with the consideration, that the least value which each plane angle, one with another, could have, would be the angle of an equilateral triangle, six of which angles are equal to four right angles; and, consequently, greater than the sum of the plane angles in any solid angle whatever.

193. Prop. *If* m = *mean number of plane angles in a solid angle; then, generally,*

$$\Sigma \{2(m+n) - m\,n\}\, f_n = 4\,m.$$

$$\text{For, } \frac{2E}{S} = \frac{2\,\Sigma\, n\, f_n}{4+\Sigma(n-2)f_n} = m, \quad \text{(art. 185.)}$$

$$\therefore\ 2\,\Sigma\, n\, f_n = 4\,m + \Sigma\,(m\,n - 2\,m)f_n,$$

hence, $\Sigma \{2(m+n) - m\,n\}\, f_n = 4\,m.$

194. Cor. By means of this equation, the value or values of n and of f_n being assigned, m is easily obtained; thus, if the polyhedron be a regular tetrahedron, $n=3$, and $f_3 = 4 = \text{F}$,

$$\therefore\ (6 - m)\,4 = 4\,m, \text{ and } m = 3.$$

If the values of n be 3 and 4,

$$m = \frac{6\,f_3 + 8\,f_4}{4 + f_3 + 2\,f_4}.$$

195. Prop. *If a polyhedron has* all *its faces triangular, and if its solid angles be formed by five and six plane angles, the solid angles formed by five plane angles will always amount to* 12, *whilst those formed by six plane angles may be any number whatever.*

Let m_5 and m_6 express the number of solid angles formed respectively by five and six plane angles,

then, $\text{S} = m_5 + m_6$, and $6\,\text{S} = 6\,m_5 + 6\,m_6$,

and $2\,\text{E} = 5\,m_5 + 6\,m_6$,

hence, $6\,\text{S} - 2\,\text{E} = m_5$;

but $2\,\text{E} = \Sigma\, n\, f_n = 3\,\text{F}$, since the only value of $n = 3$;

and $2\,\text{S} = 4 + \Sigma\,(n-2)f_n = 4 + \text{F}$,

$\therefore\ 6\,\text{S} = 12 + 3\,\text{F}$,

hence, $6\text{S} - 2\text{E} = 12 = m_5$:

hence, also, $S = 12 + m_6, \qquad (1.)$

$$\therefore\ 2\,S = 24 + 2\,m_6 = 4 + F,$$

$$\therefore\quad F = 20 + 2\,m_6, \qquad (2.)$$

$$\text{and } 2\,E = 3\,F = 60 + 6\,m_6,$$

$$\therefore\quad E = 30 + 3\,m_6. \qquad (3.)$$

The equations (1.), (2.), (3.), show that m_6 being indeterminate, S, F, and E, are likewise indeterminate.*

---

* *Legendre's* solution of the problem, ***To find the number of conditions necessary for determining any polyhedron.***

"Suppose, first, that the polyhedron is ***of a determinate kind***, in other words, that we know the number of its faces, the number of their sides individually, and their arrangement with regard to one another. We therefore know the numbers, F, S, E, and likewise f_3, f_4, f_5, f_6, &c.; we only want farther to discover the actual number of given quantities, lines or angles, by means of which the polyhedron may be constructed and determined.

"Let us examine one of the polyhedron's faces, which we shall regard as its base. Suppose n to be the number of its sides; there will be $2n - 3$ data required to determine this base. The solid angles out of this base amount in number to $S - n$: the vertex of each solid angle requires three data for determining it; hence the position of $S - n$ vertices will require $3S - 3n$; to which adding the $2n - 3$ data of the base, we shall have in all $3S - n - 3$. But this number in general is too great; it must be diminished by the number of conditions necessary for making the vertices which correspond to the same face lie all in one plane. We have called the number of sides in the base n; let us in like manner call the number of sides in the other faces n', n'', &c. Three points determine a plane; hence whatever more than 3 are found in each of the numbers n', n'', &c., will give just so many conditions for making the different vertices lie in the planes of the faces to which they belong; and the total number of conditions will be equal to the sum $(n' - 3) + (n'' - 3) + (n''' - 3) +$ &c. But the number of terms in this series is $F - 1$; and, moreover, $n + n' + n'' +$ &c. $= 2\,E$: hence the sum of the series will be $2\,E - n - 3\,(F - 1)$. From this sum take away $3\,S - n - 3$; there will remain $3\,S - 2\,E + 3\,F - 6$, a quantity, which by reason of $S + F = E + 2$, may be reduced to E. Hence ***the number of data necessary for determining a polyhedron, among all those of the same species, is equal to the number of its edges.***

"Observe, however, that the data here spoken of must not be taken at random among the lines and angles which constitute the elements of the poly-

196. PROP. *Given the three edges* a, b, c, *of a parallelopiped meeting in a point, and* A, B, C, *the angles between the edges; then the perpendicular altitude of the parallelopiped, or*

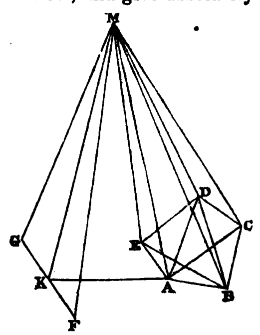
$$p = \frac{2c}{\sin A} \sqrt{\{\sin S . \sin (S-A) . \sin (S-B) . \sin (S-C)\}}.$$

Let the three edges of the parallelopiped be $OA = a$,

---

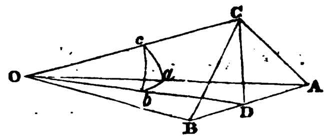
hedron; for, although there were as many equations as unknown quantities, it might happen that certain relations between the known quantities might render the problem indeterminate. Thus, from the theorem just discovered, it might seem that a knowledge of the edges alone would be enough for determining the polyhedron; yet there are cases in which this knowledge of itself is not sufficient. If, for example, any prism not triangular were given, an infinite number of other prisms might be formed having edges equal and placed in the same manner. For, whenever the base has more than three sides, the angles may be changed though the same sides are retained, and thus the base may have an infinite number of different forms; also the position of the prism's longitudinal edge with regard to the plane of the base may be changed; finally, these two changes may be combined with each other; and, from every new arrangement, a new prism will result still having its edges or sides unchanged. From all which, it is clear, that in this case the edges alone are not enough for determining the solid.

"The data which it is proper to select for determining a solid, are those which have no indeterminateness, and give absolutely only one solution. And



first, the base ABCDE will be determined by this among other modes; by knowing the side AB with the adjacent angles BAC, ABC for the point C;

$OB = b$, $OC = c$, meeting in O; from C draw CD perpendicular to the plane AOB, and join OD. With centre O



and rad $= 1$, let a spherical surface be described cutting OA, OD, OC, in the points a, b, c, which are supposed to be

---

the angles BAD, ABD for the point D; and so for all the rest. Next, let M be a point without the base whose position it is required to determine: this point will be determined, if, imagining the pyramid MABC or simply the plane MAB, we know the angles MAB, ABM, and the inclination of the plane MAB to the base ABC. If by means of three analogous data, the position of each vertex lying without the base of the polyhedron is determined, the polyhedron, it is evident, will be absolutely determined, and so that two polyhedrons constructed with the same data must of necessity be equal; or symmetrically equal, if constructed on different sides of the plane of the base.

"It is not always required to have three data for determining each vertex of a polyhedron; for if the point M must be found in a plane already determined, whose intersection with the base is FG, it will be sufficient, after having assumed FG at will, if we know the angles MGF, MFG; and thus one datum less will be enough. If the point M must be found in two planes already determined, or in their common intersection MK, which meets ABC in K, we shall in this case already know the side AK, the angle AKM, and the inclination of the plane AKM to the base; hence it will be enough to have for a new datum the angle MAK. By such means, the number of data necessary for determining a polyhedron absolutely and without any ambiguity will always be reduced to E, the number of its edges.

"The side AB and a number $E - 1$ of given angles determine a polyhedron; another side assumed at pleasure and the same angles determined a similar polyhedron. Hence it follows that *the number of conditions necessary for determining the similarity of two polyhedrons belonging to the same species is equal to the number of edges* minus *one*.

"The question we have just resolved would be much simpler, if, instead of knowing the species of the polyhedron, we knew only S the number of its solid

connected by arcs of great circles so as to form a spherical triangle having the angle abc equal to a right angle, because the plane COD is perpendicular to the plane AOB:

then, $\sin bc = \sin ac.\ \sin bac$, by *Napier's* rules;

or, $\sin \text{COD} = \sin \text{AOC}.\ \sin bac$;

let the angle $\text{AOB} = \text{A}, \text{AOC}=\text{B}, \text{BOC}=\text{C}$, and $\text{S} = \frac{\text{A}+\text{B}+\text{C}}{2}$;

hence, $\sin \text{COD} = \sin \text{B} \frac{2}{\sin \text{B}.\sin \text{A}} \sqrt{\{\sin \text{S}.(\text{S}-\text{A}).\sin(\text{S}-\text{B}).\sin(\text{S}-\text{C})\}}$,

$$\therefore\ \text{CD} = c.\ \sin \text{COD},$$

and $p = \frac{2c}{\sin \text{A}} \sqrt{\{\sin \text{S}.\ \sin(\text{S}-\text{A}).\ \sin(\text{S}-\text{B}).\ \sin(\text{S}-\text{C})\}}$.

197. **Prop.** *The whole surface of the parallelopiped* $= 2\,(\text{ab}.\ \textit{sin}\ \text{A} + \text{ac}.\ \textit{sin}\ \text{B} + \text{bc}.\ \textit{sin}\ \text{C})$.

For twice the area of the triangle $\text{AOB} = ab \sin \text{A}$, hence $2\,ab \sin \text{A} =$ surface of two parallel sides or faces; and the same being true for the remaining faces in terms of their edges and the included angles, it follows that the whole surface

$$= 2\,(a\,b.\ \sin \text{A} + a\,c.\ \sin \text{B} + b\,c.\ \sin \text{C}).$$

---

angles. In that case, determine three vertices at pleasure by means of a triangle in which are three data; this triangle will be regarded as the base of the solid; then the number of vertices out of this base will be S — 3; and since the determination of each of them requires three data, the total number of data necessary for determining the polyhedron will evidently be 3 + 3 (S — 3), or 3 S — 6.

"Hence 3 S — 7 conditions will be necessary for determining the similarity of two polygons having the same number S of solid angles."

198. PROP. *The content of the parallelopiped*
$= 2\,a\,b\,c\,\sqrt{}\{\sin \text{S}. \sin(\text{S}-\text{A}). \sin(\text{S}-\text{B}). \sin(\text{S}-\text{C})\}$.

For the content
$=$ area of base $\times\ p$,
$=$ OA. OB. sin AOB $\times\ p$,
$= ab. \sin \text{A} \times \dfrac{2c}{\sin \text{A}} \sqrt{}\{\sin \text{S}. \sin(\text{S}-\text{A}). \sin(\text{S}-\text{B}). \sin(\text{S}-\text{C})\}$,
$= 2\,a\,b\,c\,\sqrt{}\{\sin \text{S}. \sin(\text{S}-\text{A}). \sin(\text{S}-\text{B}). \sin(\text{S}-\text{C})\}$.

199. PROP. *The diagonal of the parallelopiped, or*
$\text{D} = \sqrt{}(a^2+b^2+c^2+2\,ab. \cos \text{A}+2\,ac. \cos \text{B}+2\,bc. \cos \text{C})$.

Let $d = 2\,\text{OD}$ be the diagonal of the face which
$= 2\,\text{AOB}$; then
$d^2 = a^2 + b^2 + 2\,a\,b. \cos \text{A}$:
and, similarly,
$\text{D}^2 = d^2+c^2+2\,c\,d. \cos \text{COD}$,
$= a^2+b^2+c^2+2\,a\,b. \cos \text{A}+2\,c\,d. \cos \text{COD}$;
now,
$\cos\text{COD} = \cos bc = \cos ab. \cos ac + \sin ab. \sin ac. \cos bac$,
$= \cos \text{B}. \cos ab + \sin \text{B}. \sin ab. \left(\dfrac{\cos \text{C} - \cos \text{A}. \cos \text{B}}{\sin \text{A}. \sin \text{B}}\right)$,
$= \dfrac{1}{\sin \text{A}}(\sin \text{A}. \cos \text{B}. \cos ab + \cos \text{C}. \sin ab - \cos \text{A}. \cos \text{B}. \sin ab)$,
$= \dfrac{1}{\sin \text{A}}\{\cos \text{C}. \sin ab + \cos \text{B}. \sin(\text{AOB} - a\text{O}b)\}$,
$= \dfrac{1}{\sin \text{A}}(\cos \text{C}. \sin \text{AOD} + \cos \text{B}. \sin \text{BOD})$;
but, in the triangles, the sides of which are a, d, and b, d,
$\dfrac{a}{d} = \dfrac{\sin \text{BOD}}{\sin \text{A}}$, and $\dfrac{b}{d} = \dfrac{\sin \text{AOD}}{\sin \text{A}}$;

hence, $\cos \text{COD} = \frac{a}{d}. \cos \text{B} + \frac{b}{d}. \cos \text{C}$,

$\therefore \text{D} = \sqrt{(a^2+b^2+c^2+2ab.\cos\text{A}+2ac.\cos\text{B}+2bc.\cos\text{C})}$.

200. Cor. From this value of D, it is easily shown, by changing the signs of cos A, cos B, cos C, properly for the other three diagonals, that the sum of the squares of the four diagonals of every parallelopiped $= 4a^2+4b^2+4c^2$.

201. Prop. *If* a, b, c, *be the edges of a triangular pyramid which meet in a point, and* A, B, C, *be the angles between the edges; then the content*

$= \frac{1}{3}\, abc \sqrt{\{\sin \text{S}. \sin(\text{S}-\text{A}). \sin(\text{S}-\text{B}). \sin(\text{S}-\text{C})\}}$.

Let AOB be the base of the pyramid, and let its perpendicular altitude be found, as in (art. 196.), then, since

$\text{AOB} = \frac{1}{2}\, ab. \sin \text{A}$

and $p = \frac{2c}{\sin\text{A}}\sqrt{\{\sin \text{S}. \sin(\text{S}-\text{A}). \sin(\text{S}-\text{B}). \sin(\text{S}-\text{C})\}}$;

the content of the pyramid, which

$= \frac{1}{3}\text{AOB} \times p = \frac{abc}{3}\sqrt{\{\sin\text{S}.\sin(\text{S}-\text{A}).\sin(\text{S}-\text{B}).\sin(\text{S}-\text{C})\}}$.

Or, since the pyramid $= \frac{1}{3}$ of prism, (Euc. XII. 7), it $= \frac{1}{6}$ of the parallelopiped described with the same edges.

202. The surface of the *three* sides, of which the edges are given $= \frac{1}{2}\,(ab. \sin \text{A} + ac. \sin \text{B} + bc. \sin \text{C})$.

203. Prop. *If* e, f, g, *denote the remaining edges of the pyramid opposite to the angles* A, B, C; *its content*

$= \frac{1}{12}\sqrt{(4\,a^2b^2c^2 - a^2\text{L}^2 - b^2\text{K}^2 - c^2\text{H}^2 + \text{HKL})}$,

where $\text{H} = a^2 + b^2 - e^2$,

$\text{K} = a^2 + c^2 - f^2$,

$\text{L} = b^2 + c^2 - g^2$.

For, by attending to the last figure,

$$\cos A = \frac{H}{2\,ab},\ \cos B = \frac{K}{2\,ac},\ \cos C = \frac{L}{2\,bc};$$

and the content has been found

$$=\frac{abc}{3}\sqrt{\{\sin S.\ \sin(S-A).\ \sin(S-B).\ \sin(S-C)\}},$$

$$=\frac{abc}{6}\sqrt{\{1-(\cos A)^2-(\cos B)^2-(\cos C)^2+2\cos A.\cos B.\cos C\}},$$

$$=\frac{abc}{6}\sqrt{\left\{1-\left(\frac{H}{2\,ab}\right)^2-\left(\frac{K}{2\,ac}\right)^2-\left(\frac{L}{2\,bc}\right)^2+\frac{HKL}{(2\,abc)^2}\right\}},$$

$$=\tfrac{1}{12}\sqrt{(4\,a^2b^2c^2-a^2L^2-b^2K^2-c^2H^2+HKL)}.$$

The radii of the spheres described in, and circumscribed about a pyramid, may be expressed in terms of the above data; but it is scarcely necessary to discuss these two problems in this place: it will be sufficient to refer the student to *Legendre's* Geom. and Trigon., note 5, prob. 7. and 8.

APPENDIX.

ON LOGARITHMS.

1. In the solution of trigonometrical problems, the common rules of arithmetic are adequate to almost every case. But the arithmetic operations would be often so laborious, as to be only just within the limits of human industry. In the infancy of mathematics, this difficulty would not have been so severely felt; but, when scientific men began to turn their attention to subjects of natural philosophy, they found their progress impeded by the tedious calculations which were absolutely necessary to be performed. Under this embarrassment, they naturally turned their thoughts to the invention of some means of evading the difficulty. After several essays, John Napier, a Scotch Baron, hit upon the method of Logarithms. His invention received a most important improvement from Henry Briggs, a Fellow of St. John's College, Cambridge, and Professor of Geometry in the University of Oxford. To explain the nature of these Logarithms, and how to calculate them, is the object of this treatise. The introductions to the tables will afford the student rules for using them and examples of their utility.

2. DEF. *If* u $= a^x$, x *is called the logarithm of* u *to the base* a, *and is thus denoted, log_a* u.*

3. By assuming different values of the base, there will be as many different systems of logarithms. No systems, however, are used, except the Napierien and Briggean. The base of the former is

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots = 2.7182818\ldots\ldots$$

and is always called e; that of the latter is 10. The Briggean, from their nature, are often called *decimal;* and *tabular*, because they are used in arithmetic operations, and therefore inserted in the tables. They are of such frequent use, that they are denoted simply by log, instead of $\log_{10}$.

4. PROP. *In every system the logarithm of the base is unity.*

$$\text{For, } u = a^{\log_a u}, \quad (2.)$$
$$\therefore a = a^{\log_a a},$$
$$\therefore \log_a = 1.$$

5. PROP. *In every system the logarithm of unity is zero.*

$$\text{For, } 1 = \frac{a}{a} = a^{1-1} = a^0,$$
$$\therefore \log_a 1 = 0.$$

6. COR. The logarithm is positive, or negative, according as the number is greater or less than unity.

---

* Mathematicians are indebted for this notation to Mr. Jarrett, of Catherine Hall, Cambridge.

7. PROP. ***In every system the logarithm of zero is an infinitely great quantity, negative, or positive, according as the base is greater or less than unity.***

$$\text{For, } 0 = \frac{1}{\infty} = \frac{1}{a^{\infty}}, \text{ (if } a > 1),$$
$$= a^{-\infty},$$
$$\therefore \log_a 0 = -\infty.$$

$$\text{If } a < 1,$$
$$0 = a^{\infty},$$
$$\therefore \log_a 0 = \infty.$$

8. PROP.
$$Log_a(u_1.u_2.u_3...u_n) = log_a u_1 + log_a u_2 + log_a u_3 + \ldots + log_a u_n.$$

For, by the definition of a logarithm,

$$u_1 = a^{\log_a u_1},$$
$$u_2 = a^{\log_a u_2},$$
$$u_3 = a^{\log_a u_3},$$
$$\ldots = \ldots.$$
$$u_n = a^{\log_a u_n},$$
$$\therefore u_1.\, u_2.\, u_3 \ldots\ldots u_n = a^{(\log_a u_1 + \log_a u_2 + \log_a u_3 + \cdots + \log_a u_n)}.$$
$$\therefore \log_a(u_1.u_2.u_3....u_n) = \log_a u_1 + \log_a u_2 + \log_a u_3 + .. + \log_a u_n.$$

9. PROP. $Log_a \dfrac{u_1}{u_2} = log_a u_1 - log_a u_2.$

$$\text{For, } u_1 = a^{\log_a u_1},$$
$$u_2 = a^{\log_a u_2},$$

$$\therefore \frac{u_1}{u_2} = \frac{a^{\log_a u_1}}{a^{\log_a u_2}} = a^{(\log_a u_1 - \log_a u_2)},$$

$$\therefore \log_a \frac{u_1}{u_2} = \log_a u_1 - \log_a u_2.$$

10. PROP. *$Log_a (u^{\frac{m}{n}}) = \frac{m}{n} \cdot log_a u.$*

$$\text{For, } u = a^{\log_a u},$$

$$\therefore u^{\frac{m}{n}} = a^{\frac{m}{n} \log_a u},$$

$$\therefore \log_a u^{\frac{m}{n}} = \frac{m}{n} \log_a u.$$

11. PROB. *To explain the particular advantages of the decimal logarithms.*

In this system, the logarithm of 10 is unity. Hence, the logarithms of numbers between 10 and 0 are fractions less than unity. These are placed in the tables. And, since $\log 10^m u = \log 10^m + \log u = m + \log u$, the logarithms of all other numbers may be found by the addition or subtraction of the quantity m, which is called the *characteristic*. For example, the tables give

$$\begin{aligned} \log 1.6681 &= .2222221, \\ \therefore \log 16681 &= 4.2222221, \\ \log 1668.1 &= 3.2222221, \\ \log 166.81 &= 2.2222221, \\ \log 16.681 &= 1.2222221, \\ \log 1.6681 &= .2222221, \\ \log .16681 &= \bar{1}.2222221, \\ \log .016681 &= \bar{2}.2222221, \\ \ldots\ldots &= \ldots\ldots \end{aligned}$$

This would not be the case if any other number were the base, for then the logarithm of 10^m would not be m. The finding other logarithms of numbers would thus become a laborious operation, unless the tables were increased in size, by inserting the separate logarithms. And for the inverse operation of finding the number from the logarithm this increase would evidently become absolutely necessary. It is, therefore, chiefly for the latter reason that *Briggs'* alteration of the base from e to 10 is so great an improvement.

12. PROB.

$$Log_a\, u = \frac{(u-1)-\frac{1}{2}(u-1)^2+\frac{1}{3}(u-1)^3-\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots}.$$

For, let $\log_a u = x$,

$$\therefore\ u = a^x,$$

$$\therefore\ u^y = a^{xy},$$

$$\therefore\ \frac{u^y-1}{y} = \frac{a^{xy}-1}{y},$$

$$\therefore\ \frac{\{1+(u-1)\}^y-1}{y} = \frac{\{1+(a-1)\}^{xy}-1}{y}.$$

Therefore, by the binomial theorem,

$$(u-1)+\frac{(y-1)}{1.\ 2}(u-1)^2+\frac{(y-1)(y-2)}{1.\ 2.\ 3}(u-1)^3+\ldots=,$$

$$x\left\{(a-1)+\frac{(xy-1)}{1.\ 2}(a-1)^2+\frac{(xy-1)(xy-2)}{1.\ 2.\ 3}(a-1)^3+\ldots\right\}$$

This will obtain whatever be the value of y. Let $y = 0$, therefore, by division,

$$\log_a u = x = \frac{(u-1)-\frac{1}{2}(u-1)^2+\frac{1}{3}(u-1)^3-\ldots}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\ldots}.$$

13. COR. 1. In the *Napierien* logarithms, the base, which is always called e, is assumed of such a value, that

$$(e-1)-\frac{1}{2}(e-1)^2+\frac{1}{3}(e-1)^3-\ldots = 1,$$

$$\therefore\ \log_e u = (u-1)-\frac{1}{2}(u-1)^2+\frac{1}{3}(u-1)^3-\ldots$$

14. Cor. 2. $\text{Log}_e\, u^{\frac{1}{m}} = \frac{1}{m} \log_e u,$

$$\therefore \log_e u = m \log_e (u^{\frac{1}{m}}),$$

$$= (u^{\frac{1}{m}} - 1) - \tfrac{1}{2}(u^{\frac{1}{m}} - 1)^2 + \tfrac{1}{3}(u^{\frac{1}{m}} - 1)^3 - ..$$

15. Cor. 3. $\text{Log}_e (1+u) = u - \frac{1}{2}u^2 + \frac{1}{3}u^3 - \ldots$

16. Cor. 4. $\text{Log}_e\, a = (a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - ..$

$$\therefore \log_a u = \frac{\log_e u}{\log_e a}.$$

The quantity $\frac{1}{\log_e a}$ by which the *Napierien* logarithm is multiplied to obtain the logarithm to the base a, is called the modulus of that system. Hence the modulus of the decimal logarithms is $\frac{1}{\log_e 10}$, and is always called M.

17. Prop. $a^x = 1 + \frac{x \log_e a}{1} + \frac{x^2 (\log_e a)^2}{1.\ 2.\ 3} + \ldots$

Let $a^x = 1 + Ax + Bx^2 + Cx^3 + \ldots.$

The first term is assumed unity, because that is the value of a^x when x is zero.

$$\therefore \quad a^y = 1 + Ay^2 + By^2 + Cy^2 + \ldots.$$

$$\therefore \quad (a^y)^{\frac{x}{y}} = \{1 + y\,(A + By + Cy^2 + \ldots)\}^{\frac{x}{y}}$$

$$\therefore \quad a^x = 1 + x\,(A + By + Cy^2 + \ldots)$$

$$+ \frac{x.\,(x-y)}{1.\ 2}(A + By + Cy^2 + \ldots)^2$$

$$+ \frac{x.\,(x-y).\,(x-2y)}{1.\ 2.\ 3}(A + By + Cy^2 + \ldots)^3,$$

$$+ \ldots.$$

And this must obtain whatever be the value of y. Let $y = 0$.

$$\therefore\ a^x = 1 + Ax + \frac{A^2x^2}{1.\ 2} + \frac{A^3x^3}{1.\ 2.\ 3} + \dots$$

But, by the binomial theorem,

$$\begin{aligned} a^x &= \{1+(a-1)\}^x, \\ &= 1 + x\,(a-1) + \frac{x.\,(x-1)}{1.\ 2}\,(a-1)^2 + \dots \\ &= 1 + x\,\{(a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \dots\} \\ &\qquad + \text{ terms involving } x^2,\ x^3, \text{ \&c.} \end{aligned}$$

$$\begin{aligned} \therefore\ A &= (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \dots \\ &= \log_e a, \ (13.) \end{aligned}$$

$$\therefore\ a^x = 1 + \frac{x.\ \log_e a}{1} + \frac{x^2(\log_e a)^2}{1.\ 2} + \frac{x^3(\log_e a)^3}{1.\ 2.\ 3} + \dots$$

18. Cor. 1. By substituting e for a,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{1.\ 2} + \frac{x^3}{1.\ 2.\ 3} + \dots$$

19. Cor. 2. By making $x = 1$,

$$\begin{aligned} e &= 1 + \frac{1}{1} + \frac{1}{1.\ 2} + \frac{1}{1.\ 2.\ 3} + \dots \\ &= 2.7182818.\ \dots \end{aligned}$$

20. Prop. $Log\ \dfrac{1+m}{1-m} = 2M\left\{m + \dfrac{m^3}{3} + \dfrac{m^5}{5} + \dots\right\}$.

For, by articles 12 and 16,

$$\log(1+m) = M\left\{\ m - \frac{m^2}{2} + \frac{m^3}{3} - \dots\right\},$$

$$\therefore\ \log(1-m) = M\left\{-m - \frac{m^2}{2} - \frac{m^3}{3} - \dots\right\}.$$

Therefore, by subtraction, (9.)

$$\text{Log}\,\frac{1+m}{1-m} = 2\,\text{M}\left\{m + \frac{m^3}{3} + \frac{m^5}{5} + \ldots\right\}.$$

21. COR. 1. By making $\frac{1+m}{1-m} = \frac{u}{v}$, $m = \frac{u-v}{u+v}$,

$$\therefore \log u - \log v = 2\,\text{M}\left\{\left(\frac{u-v}{u+v}\right) + \tfrac{1}{3}\left(\frac{u-v}{u+v}\right)^3 + \ldots\right\}$$

22. COR. 2. By making $\frac{1+m}{1-m} = 1 + \frac{u}{v}$, $m = \frac{v}{2u+v}$.

$$\therefore \log(u+v) - \log u = 2\,\text{M}\left\{\frac{v}{2u+v} + \tfrac{1}{3}\left(\frac{v}{2u+v}\right)^3 + \ldots\right\}.$$

23. COR. 3.

By making $\frac{1+m}{1-m} = \frac{u^2}{u^2-v^2} = \frac{u^2}{(u+v).\ (u-v)}$,

$$m = \frac{v^2}{2\,u^2+v^2},$$

$$\therefore \log(u+v) = 2\log u - \log(u-v),$$

$$-\,2\,\text{M}\left\{\left(\frac{v^2}{2\,u^2+v^2}\right) + \tfrac{1}{3}\left(\frac{v^2}{2\,u^2+v^2}\right)^3 + \ldots\right\}.$$

24. COR. 4.

By making $\frac{1+m}{1-m} = \frac{(u-v)^2\,(u+2\,v)}{(u+v)^2\,(u-2\,v)}$,

$$= \frac{u^3-3\,u\,v^2+2\,v^3}{u^3-3\,u\,v^2-2\,v^3},$$

$$\log(u+2\,v) = 2\log(u+v) + \log(u-2\,v) - 2\log(u-v),$$

$$+\,2\,\text{M}\left\{\left(\frac{2\,v^3}{u^3-3\,u\,v^2}\right) + \tfrac{1}{3}\left(\frac{2\,v^3}{u^3-3\,u\,v^2}\right)^3 + \ldots\right\}.$$

25. COR. 5.

By making $\frac{1+m}{1+m} = \frac{u^2\,(u+7)^2\,(u-7)^2}{(u^2-3^2)(u^2-5^2)(u^2-8^2)}$,

$$= \frac{u^6 - 98\,u^4 + 2401\,u^2}{u^6 - 98\,u^4 + 2401\,u^2 - 14400},$$

$$\begin{aligned} \log(u+8) = {} & 2\log(u+7) - \log(u+5) - \log(u+3), \\ & + 2\log u - \log(u-3) - \log(u-5), \\ & + 2\log(u-7) - \log(u-8), \\ & - 2\,\mathrm{M}\left\{\frac{7200}{u^6 - 98\,u^4 + 2401\,u^2 - 7200} + \ldots\right\}. \end{aligned}$$

26. **Prob.** *To calculate the arithmetic value of the modulus of the decimal logarithms.*

By article 14,

$$\log_e u = m\,\{(u^{\frac{1}{m}} - 1) - \tfrac{1}{2}(u^{\frac{1}{m}} - 1)^2 + \tfrac{1}{3}(u^{\frac{1}{m}} - 1)^3 - \ldots\}$$

Now, since $u^{\frac{1}{\infty}} = u^0 = 1$, therefore, whatever be the value of u, $u^{\frac{1}{m}}$ may be made to approach to unity, to any degree of proximity, by taking m of sufficient magnitude; $u^{\frac{1}{m}}$ being greater or less than unity, according as u is greater or less than unity. Let $u = 10$, and $m = 2^{54}$. Then $u^{\frac{1}{m}}$ will be obtained by extracting the square root of 10 fifty-four times, and the result will be found to exceed unity by a decimal with fifteen cyphers after the decimal point. And $\frac{1}{2}(u^{\frac{1}{m}} - 1)^2$ will be found to have thirty-one, and $\frac{m}{2}(u^{\frac{1}{m}} - 1)^2$ fifteen cyphers after the decimal point; therefore, $m\,(u^{\frac{1}{m}} - 1)$ will give $\log_e 10$, and therefore, $\frac{1}{\log_e 10}$, accurately, to fifteen decimal places.

27. Prop. *To construct a table of the logarithms of numbers.*

The *Napierien* logarithms of all *prime* numbers must be calculated by the series in articles 21 .... 25. All other numbers must be separated into their prime factors, and their logarithms must be added together. The *Napierien* logarithms having been found, they must be multiplied by the modulus $\frac{1}{\log_e 10}$, and the decimal logarithms will be obtained. The following examples will serve as illustrations.

In 21, let $u = 2$, and $v = 1$;

$$\therefore \log 2 = 2\,\mathrm{M}\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \ldots\right\}.$$

Let $u = 3$, and $v = 1$;

$$\therefore \log 3 = \mathrm{M}\left\{\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4^3} + \frac{1}{5}\cdot\frac{1}{4^5} + \ldots\right\},$$

$$\log 4 = \log(2^2) = 2\log 2,$$

$$\log 5 = \log\frac{10}{2} = \log 10 - \log 2 = 1 - \log 2,$$

$$\log 6 = \log 2 + \log 3.$$

In art. 24, let $u = 5$, and $v = 1$;

$$\therefore \log 7 = 2\log 6 + \log 3 - 2\log 4,$$
$$+ 2\,\mathrm{M}\left\{\frac{1}{30} + \frac{1}{3}\cdot\frac{1}{30^3} + \frac{1}{5}\cdot\frac{1}{30^5} + \ldots\right\},$$

$$\log 8 = 3\log 2,$$
$$\log 9 = 2\log 3,$$
$$\log 10 = 1.$$

In art. 24, let $u = 9$, and $v = 1$;

$$\log 11 = 2 + \log 7 - 2\log 8,$$
$$+ 2\,\mathrm{M}\left\{\frac{1}{244} + \frac{1}{3}\cdot\frac{1}{244^3} + \frac{1}{5}\cdot\frac{1}{244^5} + \ldots\right\}.$$

28. PROB. *To find the logarithm of a number consisting of seven digits.*

Let v = the given number,
u = the number consisting of the first five digits,
h = the sixth, and k = the seventh digit,
$\therefore\ v = 100\,u + 10\,h + k$,

$$\log v - \log 100\,u = \log \frac{100\,u - 10\,h + k}{100\,u}$$

$$= \log \left\{1 + \frac{1}{u}\left(\frac{h}{10} + \frac{k}{100}\right)\right\},$$

$$= \text{M} \left\{\frac{1}{u}\left(\frac{h}{10} + \frac{k}{100}\right) - \frac{1}{2\,u^2} \cdot \left(\frac{h}{10} + \frac{k}{100}\right)^2 + \ldots\right\}.$$

Similarly,

$$\log (u+1) - \log u = \text{M} \left\{\frac{1}{u} - \frac{1}{2\,u^2} + \ldots\right\}.$$

Now, since u is a number of five digits, its least value is 10000, and therefore the greatest value of u is $\frac{1}{10000}$, and the greatest value of $\frac{\text{M}}{2\,u^2} = \frac{.434294481}{2\,(10000)^2} = .00000000\,34\ldots$ $\frac{\text{M}}{2\,u^2}$ will have no significant value, and since $\frac{h}{10} + \frac{k}{100} > 1$, $\frac{\text{M}}{2\,u^2}\left(\frac{h}{10} + \frac{k}{100}\right)^2$ may also be omitted without affecting the accuracy of the result to seven places of decimals. And a portion of the remaining terms in the two series may be neglected, therefore

$$\log v - \log 100\,u = \frac{\text{M}}{u}\left(\frac{h}{10} + \frac{k}{100}\right),$$

$$\log (u+1) - \log u = \frac{\text{M}}{u},$$

$$\therefore \frac{\log v - \log 100\,u}{\log(u+1) - \log u} = \frac{h}{10} + \frac{k}{100},$$

$$\therefore \log v = 2 + \log u + \left(\frac{h}{10} + \frac{k}{100}\right)\{\log(u+1) - \log u\}.$$

29. COR. 1. In the small variations of any number, the increments of the logarithm are proportional to the increments of the number.

Ex. To find log 4352164.

$$\text{Here } u = 43521,\ h = 6, \text{ and } k = 4.$$

$$\text{Log}(u+1) = 4.\ 6387088,$$

$$\log u = 4.\ 6386989,$$

$$\therefore \log(u+1) - \log u = .0000099,$$

$$= \text{D, suppose.}$$

$$h\frac{\text{D}}{10} = .00000594,$$

$$k\frac{\text{D}}{100} = .000000396,$$

$$\therefore \log v = 6.\ 6387052\ 36,$$

$$= 6.\ 6387052,$$

by rejecting the decimals after the seventh place.

30. COR. 2. The labor of multiplication in every case is obviated by the construction of small tables, called *proportional parts*, for all values D, h and k. But separate tables for h and k are unnecessary, since the only difference between them is, that the latter are one step farther in the decimal places. The following example will serve as an illustration.

Ex. To construct a table of parts proportional for all values of h when D = .0000099.

$$\therefore \frac{\text{D}}{10} = .00000099,$$

$$2\frac{\text{D}}{10} = .00000198,$$

$$3\,\frac{D}{10} = .00000297,$$
$$4\,\frac{D}{10} = .00000396,$$
$$5\,\frac{D}{10} = .00000495,$$
$$6\,\frac{D}{10} = .00000594,$$
$$7\,\frac{D}{10} = .00000693,$$
$$8\,\frac{D}{10} = .00000792,$$
$$9\,\frac{D}{10} = .00000891.$$

Or, by omitting the cyphers, and taking the nearest value of $h\,\frac{D}{10}$ to seven decimal places,

$$D = 99,$$
$$h = \left\{\begin{array}{l} 1 \;.\;.\;.\; 10 \\ 2 \;.\;.\;.\; 20 \\ 3 \;.\;.\;.\; 30 \\ 4 \;.\;.\;.\; 40 \\ 5 \;.\;.\;.\; 50 \\ 6 \;.\;.\;.\; 59 \\ 7 \;.\;.\;.\; 69 \\ 8 \;.\;.\;.\; 79 \\ 9 \;.\;.\;.\; 89 \end{array}\right\} = h\,\frac{D}{10}.$$

To find log 4352164,

$$\begin{array}{rrr} & \log 4352100 = & 6.\ 6386989, \\ \left.\begin{array}{r}\text{increments of} \\ \text{logarithm for}\end{array}\right\} & \begin{array}{r} 60 = \\ 4 = \end{array} & \begin{array}{r} 59, \\ 4, \end{array} \\ \hline \therefore & \log 4352164 = & 6.\ 6387052. \end{array}$$

31. PROB. *A logarithm being given, which is not to be found exactly in the table, to find the corresponding number.*

This may be done, by taking the number corresponding to the logarithm next inferior to the given logarithm, and calculating for the difference of the two logarithms by the table of proportional parts which gives the increment of the number corresponding to the increment of the logarithm. An example will illustrate the rule.

Ex. To find the number corresponding to the logarithm 4.6387054.

$$\begin{array}{rr}
 & 4.\ 6387054 \\
\log 43521 = & 4.\ 6386989 \\
\hline
 & 65 \\
6 \ldots\ldots\ldots & 59 \\
\hline
 & 60 \\
6 \ldots\ldots\ldots & 59 \\
\hline
 & 1
\end{array}$$

Therefore, the number is 43521. 66.

32. The problem may be solved without the proportional parts, thus,

Let v = the number whose logarithm is given,

u = the number whose logarithm is next inferior to $\log v$.

Therefore, by art. 29.

$$\frac{\log v - \log u}{\log(u+1) - \log u} = \frac{v-u}{1},$$

$$\therefore\ v = u + \frac{\log v - \log u}{\log(u+1) - \log u}.$$

Thus, in the preceding example,

$$v = 43521 + \frac{65}{99},$$
$$= 43521.66.$$

33. PROB. *To explain the tables of logarithmic trigonometric functions.*

Since many trigonometrical functions of angles are less than unity, their logarithms will be negative, whilst those greater than unity will have positive logarithms. Now, although the logarithm of zero is $-\infty$, yet the logarithm of the smallest trigonometric function of an angle, which ever comes to use, exceeds -10. On this account 10 is added to all these logarithms to make them positive. This alteration transforms $\log f\text{A}$ into $10 + \log f\text{A} = \log 10^{10} + \log f\text{A} = \log 10^{10}.f\text{A} = \log$ of the trigonometric function of an arc of which the radius is 10^{10}. Hence a formula to be adapted to the tables must have all the trigonometric functions of angles changed into the corresponding functions of arcs to radius 10^{10}. Thus, for example, when a is small

$$\sin a \doteq a\ (\cos a)^{\frac{1}{3}}.$$

When adapted to the tables, becomes

$$\frac{\sin a}{10^{10}} = a. \left(\frac{\cos a}{10^{10}}\right)^{\frac{1}{3}}.$$

$$\text{Log} \sin a - 10 = \log a + \tfrac{1}{3}(\log \cos a - 10),$$

or, if the arc a be reduced to seconds,

$$\log \sin \text{A} - 10 = \log \frac{\pi \text{A}}{648000} + \tfrac{1}{3}(\log \cos \text{A} - 10).$$

The arc a was not reduced to radius 10^{10}, because the change extends only to the trigonometric functions, and not to the arcs themselves.

34. PROB. *To find the logarithmic sines and cosines of angles.*

By making $a = \frac{m}{n}\,\frac{\pi}{2}$, (in art. 169. pt. I.) since $\sin a = \sin A$,

$$\sin\frac{m}{n}\,90^\circ = \frac{m}{n}\left(1-\frac{m^2}{2^2\,n^2}\right)\left(1-\frac{m^2}{4^2\,n^2}\right)\left(1-\frac{m^2}{6^2\,n^2}\right)\cdots$$

$$\therefore \log\sin\frac{m}{n}\,90^\circ = \log\frac{m\,(2\,n+m)\,(2\,n-m)\,\pi}{2^3\,n^3} + 10$$

$$+ \log\left(1-\frac{m^2}{4^2\,n^2}\right) + \log\left(1-\frac{m^2}{6^2\,n^2}\right)+\cdots$$

$$= \log\frac{m\,(2\,m+n)\,(2\,m-n)\,\pi}{2^3\,n^3} + 10,$$

$$+ M.\frac{m^2}{n^2}\left\{\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots\right\},$$

$$+ \frac{M}{2}\cdot\frac{m^4}{n^4}\left\{\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}+\ldots\right\},$$

$$= \frac{M}{3}\cdot\frac{m^6}{n^6}\left\{\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}+\ldots\right\}.$$

$$+ \ldots\ldots\ldots\ldots$$

Log $\left(1-\frac{m^2}{2^2\,n^2}\right)$ is not expanded, except when $\frac{m}{n}$ is very small, because the powers of $\frac{1}{2}$ are not sufficiently convergent to make only a few terms of the series necessary.

35. COR. 1. Similarly,

$$\log\cos\frac{m}{n}\,90^\circ = \log\frac{(n-m).\,(n+m)}{n^2} + 10,$$

$$+ M\left\{\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right\},$$

$$+ \frac{M}{2}\left\{\frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots\right\},$$

$$+ \frac{M}{2}\left\{\frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \ldots\right\},$$

$$+ \ldots\ldots\ldots$$

36. Cor. 2. The formulæ in articles 288, 293, afford formulæ of verification, by giving different values of m.

37. Cor. 3. This method is too laborious to be applied to every degree. But

$$\log \sin(A+B) - \log \sin A = \log \frac{\sin(A+B)}{\sin A},$$

$$= 2M\left\{\left(\frac{\sin(A+B)-\sin A}{\sin(A+B)+\sin A}\right) + \tfrac{1}{3}\left(\frac{\sin(A+B)-\sin A}{\sin(A+B)+\sin A}\right)^3 + \ldots\right\}.$$

Hence, if a table of natural sines be given, the logarithmic table may be then filled up for every degree.

The operation for minutes &c. may be rendered much more easy by differences, if $u = \log \sin \alpha$, $\log \sin(\alpha + n\beta)$,

$$= u + n\,\Delta u + \frac{n(n-1)}{1.2}\Delta^2 u + \frac{n.(n-1).(n-2)}{1.\,2.\,3}\Delta^3 u + \ldots$$

Now, to find Δu, $\Delta^2 u$, &c., the theorem of finite differences gives

$$\Delta^n u = \frac{\Delta^n 0^n}{1.2.3..n}\cdot\frac{d^n u}{d\alpha^n}\cdot\beta^n + \frac{\Delta^n.\,0^{n+1}}{1.2.3...(n+1)}\cdot\frac{d^{n+1}u}{d\alpha^{n+1}}\cdot\beta^{n+1} + \ldots$$

A table of $\dfrac{\Delta^n.\,0^n}{1.\,2.\,3\ldots.n}\cdot\ \dfrac{\Delta^n.\,0^{n+1}}{1.\,2\ldots(n+1)}$, &c.

must be calculated, and it will serve for all values of β.

Also, since $u = \log \sin \alpha$,

$$\frac{d\,u}{d\,\alpha} = \mathrm{M} \cot \alpha,$$

$$\frac{d^2\,u}{d\,\alpha^2} = -\,\mathrm{M}\,\{1 + (\cot \alpha)^2\},$$

$$\frac{d^3\,u}{d\,\alpha^3} = \mathrm{M}\,\{\cot \alpha + (\cot \alpha)^3\},$$

$$\ldots = \ldots\ldots\ldots$$

It is only necessary to remark, Δu, $\Delta^2\,u$, &c. must be calculated to more decimal places, according to the order of differences, on account of the varying value of the coefficient. Thus, if

$$n = 30,$$

$$\therefore\ \frac{n.\,(n-1)}{1.\ 2} = \frac{30.\ 29}{1.\ 2} = 435,$$

and $$\frac{n.\,(n-1).\,(n-2)}{1.\ 2.\ 3} = \frac{30.\ 29.\ 28}{1.\ 2.\ 3} = 4060;$$

therefore, $\Delta\,u$ must be calculated to 2, $\Delta^2 u$ to 3, and $\Delta^3\,u$ to 4 more decimal places than u. It is evident that this method applies equally to *natural* sines, cosines, &c.

38. Prop. *Log sin* A$''$ = *log* A $-\ \frac{1}{3}$ (10 $-$ *log. cos* A) $+$ 4.6855749, *when* A *is small.*

$$\text{For } \sin \alpha = \alpha - \frac{\alpha^3}{1.\ 1.\ 3} + \frac{\alpha^5}{1.\ 2.\ 3.\ 4.\ 5} - \ldots$$

$$= \alpha - \frac{\alpha^3}{6} \text{ when } \alpha \text{ is small},$$

$$= \alpha.\left(1 - \frac{\alpha^2}{6}\right),$$

$$= \alpha.\left(1 - \frac{\alpha^2}{3}\right)^{\frac{1}{2}},$$

$$= \alpha.\,(\cos \alpha)^{\frac{1}{3}},$$

$$\therefore\ A'' = \frac{\pi}{180 \times 60 \times 60}.\ A.\ (\cos A)^{\frac{1}{3}},$$

$$\therefore\ \log \sin A'' - 10 = + \log 648000,$$
$$+ \log A + \tfrac{1}{3} (\log \cos A - 10),$$

$$\therefore\ \log \sin A'' = 10.4971499 - 5.8115750,$$
$$+ \log A - \tfrac{1}{3} (10 - \log \cos A),$$
$$= 4.6855749 + \log A - \tfrac{1}{3} (10 - \log \cos A).$$

39. **Prop.**

Log tan A = 4.6855749 + *log* A + $\frac{2}{3}$ (10 − *log cos* A).

$$\text{Tan } a = \frac{\sin a}{\cos a} = \frac{a - \frac{a^3}{6}}{\cos a} = \frac{a\left(1 - \frac{a^2}{2}\right)^{\frac{1}{3}}}{\cos a},$$

$$= a\ (\cos a)^{-\frac{2}{3}},$$

$$\therefore\ \tan A'' = \frac{3.141596}{648000}\ A.\ (\cos A)^{-\frac{2}{3}}$$

$$\therefore\ \log \tan A'' - 10 = 6.6855749 + \log A + \tfrac{2}{3} (10 - \log \cos A),$$

$$\therefore\ \log \tan A'' = 4.6855749 + \log A'',$$
$$+ \tfrac{2}{3} (10 - \log \cos A'').$$

40. **Cor.** Log cot $A'' = 20 - \log \tan A''$,

$$= 15.3144251 - \log A'',$$
$$-\tfrac{2}{3} (10 - \log \cos A'').$$

41. **Prop.** *Log* A'' = *log sin* A'' + $\frac{1}{3}$ (10 − *log cos* A) + 5.3144251 − 10, *when* A *is small.*

$$\text{Sin } a = a - \frac{a^3}{6}, \text{ nearly},$$

$$\therefore\ \alpha = \sin \alpha + \frac{\alpha^3}{6},$$

$$= \sin \alpha + \frac{\left(\sin \alpha + \frac{\alpha^2}{6}\right)}{6},$$

$$= \sin \alpha + \frac{(\sin \alpha)^3}{6},$$

$$= \sin \alpha \left(1 + \frac{(\sin \alpha)^2}{6}\right),$$

$$= \sin \alpha\ (1 - \sin \alpha^2)^{-\frac{1}{6}},$$

$$= \sin \alpha\ (\cos \alpha)^{-\frac{1}{3}},$$

$$\therefore\ A = \frac{648000}{3.141596} \cdot \sin A.\ (\cos A)^{-\frac{1}{3}},$$

$$\therefore\ \log A = 5.3144251 - 10 + \log \sin A,$$
$$+ \tfrac{1}{3}\ (10 - \log \cos A).$$

42. $\text{Log } A'' = 5.3144251 - 10$
$$+ \log \tan A'' - \tfrac{2}{3}\ (10 - \log \cos A'').$$

By art. 272. pt. I.

$$\alpha = \tan \alpha - \frac{(\tan \alpha)^3}{3} \text{ nearly},$$

$$= \tan \alpha.\ \{1 + (\tan \alpha)^2\}^{-\frac{1}{3}},$$

$$= \tan \alpha.\ (\cos \alpha)^{\frac{2}{3}},$$

$$\therefore\ A = \frac{648000}{3.141596} \cdot \tan A. (\cos A)^{\frac{2}{3}},$$

$$\therefore\ \log A = 5.\ 3144251 - 10 + \log \tan A.$$
$$+ \tfrac{2}{3}\ (10 - \log \cos A).$$

43. Examples are given with the rules in the introductions to the tables of logarithms, and since every student will find it

necessary to make use of the tables, it would be superfluous to give many examples in a book like this. One example, therefore, only will be given, which will illustrate article 319, part 1.

44. PROB. *To solve the cubic equation*

$$x^3 - 3.0301\,x + 2.0301 = 0,$$

by aid of trigonometry.

$$\text{Now, } \log\sin 3\text{A} = \log r - \log 2 - \tfrac{3}{2}(\log q - \log 3) + 10. \quad (317.)$$

$$\begin{aligned}
\text{Here } q &= 3.0301,\\
r &= 2.0301,\\
\therefore \log q &= .4814570,\\
\log 3 &= .4771213,\\
\therefore \log q - \log 3 &= .0043357,\\
\therefore \tfrac{3}{2}(\log q - \log 3) &= .0065036,\\
\log r &= .3075174,\\
\log 2 &= .3010300,\\
\therefore \log r - \log 2 &= .0064874,\\
\text{and } \tfrac{3}{2}(\log q - \log 3) &= .0065036,\\
\therefore \text{difference} &= \bar{1}.9999838,\\
\therefore \log\sin 3\text{A} &= 9.9999838,\\
\therefore 3\text{A} &= 89^\circ\, 30'\, 18'', \text{ by } \textit{Sherwin's} \text{ Tables},\\
\therefore \text{A} &= 29^\circ\, 50'\, 6''.
\end{aligned}$$

$$\text{Again, } \log x = \log 2 + \log\sin\text{A} - 10 + \tfrac{1}{2}(\log q - \log 3). \quad (317.)$$

$$\begin{aligned}
\log\sin\text{A} &= 9.6967965,\\
\log 2 &= .3010300,\\
\tfrac{1}{2}(\log q - \log 3) &= .0021679,\\
\therefore \log x &= \bar{1}.9999944,\\
\therefore x &= .9999870.
\end{aligned}$$

But, by inspection, it is evident that unity is a root of the proposed cubic; and, by dividing by $x-1$, the remaining roots are contained in the quadratic

$$x^2 + x - 2.0301 = 0,$$

and therefore are 1.01 and -2.01. Therefore, the result given by trigonometry is wrong by .0000130. This arises from the circumstance that the ratio of two of the roots, 1 and 1.01, is nearly a ratio of equality, and therefore the angle 3 A is nearly a right angle, and consequently not found accurately from the sine. If there had been a greater proximity between the roots, the error would have been greater.

By articles 78 and 136 of part I. the angle may be found to sufficient exactness from the table of logarithmic tangents.

THE END.

Printed by W. Metcalfe, St. Mary's Street, Cambridge.

TABLE.

[*Referred to from* p. 173.]

Sin 9°	$= \cos 81°$	$= \frac{1}{4}\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\}$.
Sin 12°	$= \cos 78°$	$= \frac{1}{8}\{\sqrt{(10+2\sqrt{5})}-\sqrt{3}\,(\sqrt{5}-1)\}$.
Sin 15°	$= \cos 75°$	$= \frac{1}{2}\{\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}\}$.
Sin 22° 30′	$= \cos 67° 30′$	$= \frac{1}{2}\sqrt{\{1+\sqrt{\tfrac{1}{2}})}-\sqrt{(1-\sqrt{\tfrac{1}{2}})}\}$.
Sin 27°	$= \cos 63°$	$= \frac{1}{8}\sqrt{2}\,\{\sqrt{(10+2\sqrt{5})}-(\sqrt{5}-1)\}$.
Sin 36°	$= \cos 54°$	$= \frac{1}{4}\{\sqrt{(4+\sqrt{(10+2\sqrt{5})})}-\sqrt{(4-\sqrt{(10+2\sqrt{5})})}\}$.
Sin 42°	$= \cos 46°$	$= \frac{1}{8}\{\sqrt{6}\,\sqrt{(5+\sqrt{5})}-(\sqrt{5}-1)\}$.
Sin 46°	$= \cos 42°$	$= \frac{1}{8}\{\sqrt{6}\,\sqrt{(5+\sqrt{5})}+(\sqrt{5}-1)\}$.
Sin 54°	$= \cos 36°$	$= \frac{1}{4}\{\sqrt{(4+\sqrt{(10+3\sqrt{5})})}+\sqrt{(4-\sqrt{(10+2\sqrt{5})})}\}$.
Sin 63°	$= \cos 27°$	$= \frac{1}{8}\sqrt{2}\{\sqrt{(10+2\sqrt{5})}+(\sqrt{5}-1)\}$.
Sin 67° 30′	$= \cos 22° 30′$	$= \frac{1}{2}\sqrt{\{(1+\sqrt{\tfrac{1}{2}})}+\sqrt{(1-\sqrt{\tfrac{1}{2}})}\}$.
Sin 75°	$= \cos 15°$	$= \frac{1}{2}\{\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{1}{2}}\}$.
Sin 78°	$= \cos 12°$	$= \frac{1}{8}\{\sqrt{(10+2\sqrt{5})}+\sqrt{3}\,(\sqrt{5}-1)\}$.
Sin 81°	$= \cos 9°$	$= \frac{1}{4}\{\sqrt{(3+\sqrt{5})}+\sqrt{(5-\sqrt{5})}\}$.

ERRATA.

Page	line	for	read
2	12	$\frac{AT}{A'P'}$	$\frac{AT}{A'T'}$
16	25	42	44
28	2	decimal	sexagesimal
37	9	$-\frac{1}{\tan\alpha}$	$-\tan\alpha$
46	18 & 22	$\frac{\beta}{2}$	$\frac{B}{2}$
60	4	$\frac{\beta}{2}$	$\frac{B}{2}$
68	24	$(\sin\alpha)^m$	$(\cos\alpha)^m$
89	3, 5, 10	$+4;\ -1;\ \frac{m}{n}$	$+1;\ m-1;\ \frac{m}{2}$
136	4, 6, 12	$\pi 2,\ 3\pi^2;\ \frac{2^2}{\pi^2};\ \frac{\pi^4}{96}$	$\pi^2,\ 3^2\ \pi^2;\ \frac{2^4}{\pi^4};\ \frac{\pi^4}{384}$
138	2	multiply the series by $\frac{180^\circ}{\pi}$	
160	9	x^2	x^3
173	12 & 13	$27^\circ;\ 62^\circ$	$22^\circ;\ 67^\circ$
256	1 & 3	A	*a*
277	6	$\cos a$	$\cos d$
319	16 & 18	> 1; a portion of	< 1; *a fortiori*
325	1, 2, 3	$M;\ \frac{M}{2};\ \frac{M}{2}$; theorem	$M\frac{m^2}{n^2};\ \frac{M}{2}\cdot\frac{m^4}{n^4};\ \frac{M}{3}\cdot\frac{m^6}{n^6}$; theory

N.B.—P. 47, omit the first line.